

Endomorphism algebras of Kuga-Satake varieties.

Evgeny Mayanskiy

November 5, 2012

Abstract

We compute endomorphism algebras of Kuga-Satake varieties associated to $K3$ surfaces.

1 Preliminary remarks.

Let V be a \mathbb{Q} -lattice of transcendental cycles on a $K3$ surface X , $\phi: V \otimes_{\mathbb{Q}} V \rightarrow \mathbb{Q}$ the polarization of the weight 2 Hodge structure on V , $E = \text{End}_{Hdg}(V)$, $\Phi: V \otimes_E V \rightarrow E$ the hermitian or bilinear form constructed in [15], $\phi = \text{tr} \circ \Phi$.

Let $C(V)$ be the Clifford algebra of the quadratic space (V, ϕ) over \mathbb{Q} , $C^+(V)$ the even Clifford algebra and $KS(X)$ the Kuga-Satake variety of X . Here we define $KS(X)$ from the weight 2 Hodge structure on the lattice of transcendental cycles V rather than on the whole lattice of primitive cycles $H^2(X, \mathbb{Q})_{\text{prim}}$. In particular, the Kuga-Satake variety defined here is isogenous to a power of the Kuga-Satake variety defined using the whole lattice of primitive cycles (see [7], [10], §4).

We want to compute the endomorphism algebra $\text{End}(KS(X))_{\mathbb{Q}} = \text{End}_{Hdg}(C^+(V))$.

Let $Z(\Phi)$ be the \mathbb{Q} -algebraic group $\text{Res}_{E/\mathbb{Q}}(SO(V, \Phi))$, if E is a totally real field, or $\text{Res}_{E_0/\mathbb{Q}}(U(V, \Phi))$, if $E = E_0(\theta)$ is a CM-field (with the totally real subfield E_0). Recall, that according to [15], $Z(\Phi)$ is the Hodge group of the Hodge structure on V .

Let $CSpin(\phi): = \{g \in C^+(V)^* \mid gVg^{-1} \subset V\}$. Consider the vector representation $\rho: CSpin(\phi) \rightarrow GL(V)$, $g \mapsto (v \mapsto gvg^{-1})$ and the spin representation $\sigma: CSpin(\phi) \rightarrow GL(C^+(V))$, $g \mapsto (x \mapsto gx)$. Let $ZSpin(\Phi): = \{g \in CSpin(\phi) \mid \rho(g) \in Z(\Phi)\} = \rho^{-1}(Z(\Phi)) \subset CSpin(\phi)$. Note that $\rho(ZSpin(\Phi)) = Z(\Phi)$.

Lemma 1. *The Mumford-Tate group of the weight 1 Hodge structure on $C^+(V)$ is the preimage with respect to ρ of the Mumford-Tate group of the weight 2 Hodge structure on V .*

Proof: The same as Proposition 6.3 in [12]. If $h_X: S^1 \rightarrow GL(V)$ and $h_{KS(X)}: S^1 \rightarrow GL(C^+(V))$ denote the corresponding Hodge structures, then $h_X = \rho \circ \sigma^{-1} \circ h_{KS(X)}$ (as

shown in [12]). *QED*

Corollary. $End(KS(X))_{\mathbb{Q}} \cong End_{ZSpin(\Phi)}(C^+(V))$, where $ZSpin(\Phi)$ acts on $C^+(V)$ via the spin representation $\sigma|_{ZSpin(\Phi)}$.

So, if $C^+(V) = \bigoplus_j T_j^{\oplus m_j}$ is the decomposition of $\sigma|_{ZSpin(\Phi)}$ into a direct sum of irreducible (mutually non-isomorphic) representations T_j , then $End(KS(X))_{\mathbb{Q}} \cong \prod_j Mat_{m_j \times m_j}(D_j)$ as \mathbb{Q} -algebras, where $D_j = End_{CSpin(\Phi)}(T_j)$.

Let us assume that $m = dim_E V \geq 3$, if $E = E_0$ is totally real, and $m = dim_E V \geq 2$, if $E = E_0(\theta)$ is a CM-field. In the totally real case condition $m \geq 3$ is automatically satisfied for any K3 surface X (see [9] and [14]). In what follows we will often denote the field of rational numbers \mathbb{Q} by k and E_0 by L . Our approach is not invariant in the sense that we choose a basis in V which diagonalizes Φ right from the start (see Section 2).

Consider the epimorphism $\pi: CSpin(\phi) \rightarrow SO(\phi)$ of algebraic groups over \mathbb{Q} (induced by the vector representation ρ above) with fiber $ker(\pi) = \mathbb{G}_m \subset CSpin(\phi)$ and its restriction $\pi_0: Spin(\phi) \rightarrow SO(\phi)$ to the subgroup $Spin(\phi) \subset CSpin(\phi)$. Then π_0 is a double etale covering [3].

The argument above shows that the Hodge group Hdg of the Kuga-Satake structure on $C^+(V)$ satisfies inclusions:

$$Hdg \subset (\pi_0^{-1}(Z(\Phi)))^0 \cdot \mathbb{G}_m \text{ and } (\pi_0^{-1}(Z(\Phi)))^0 \subset Hdg$$

(hereafter for an algebraic group G we let G^0 denote the connected component of the identity and $Lie(G)$ the Lie algebra of G).

Hence the \mathbb{Q} -algebra

$$End_{Hdg}(C^+(V)) = End_{(\pi_0^{-1}(Z(\Phi)))^0}(C^+(V)) = End_{Lie(\pi_0^{-1}(Z(\Phi)))}(C^+(V)) = End_{Lie(Z(\Phi))}(C^+(V)).$$

Let $\mathfrak{g} = Lie(Z(\Phi))$. Then $\mathfrak{g} = Res_{E/k}(\mathfrak{so}(\Phi))$, if E is totally real, or $\mathfrak{g} = Res_{E_0/k}(\mathfrak{u}(\Phi))$, if $E = E_0(\theta)$ is a CM-field ($\theta^2 \in E_0$), where $k = \mathbb{Q}$.

Hence what we are looking for is the algebra of intertwining operators $End_{\mathfrak{g}}(C^+(V))$ of the \mathbb{Q} -linear representation of the Lie algebra \mathfrak{g} over \mathbb{Q} induced by the spin representation of $\mathfrak{so}(\phi)$ in $C^+(V)$ via the inclusion of Lie algebras $\mathfrak{g} \subset \mathfrak{so}(\phi)$ corresponding to the inclusion of the \mathbb{Q} -algebraic groups $Z(\Phi) \subset SO(\phi)$ above.

The problem of computing endomorphism algebras of Kuga-Satake varieties was addressed earlier by Bert van Geemen in papers [13] and [14]. In particular, in [13] he considered the case of the CM-field, which is quadratic over \mathbb{Q} and in [14] he considered the case of the totally real field, computed the endomorphism algebra in several special cases and made some general remarks. A different computation of the endomorphism algebra of the

Kuga-Satake variety in the totally real case was done by Ulrich Schlickewei [11].

Our solution uses the same ideas as (some of the ideas) in papers [13] and [14]. We compute the decomposition of the restriction to \mathfrak{g} of the spin representation of $\mathfrak{so}(\phi)$ into irreducible subrepresentations over a splitting field of \mathfrak{g} , and then apply Galois descent.

Our main result is Theorem 1 in Section 4 complemented by the computation of primary representations (which are the multiples of irreducible representations T_j above) and division algebras (which are the endomorphism algebras of T_j) in subsequent sections. In this text a 'primary representation' means a multiple of an irreducible representation. Some general observations regarding representations over arbitrary fields are collected in the Section 2. In Section 3 we introduce Galois-invariant Cartan subalgebras. In Section 4 we compute decompositions of representations over a splitting field. In Section 5 we construct primary representations over \mathbb{Q} whose irreducible components appear in Theorem 1. In Section 6 we compute the division algebras which are the endomorphism algebras of those irreducible components. Section 7 is devoted to examples.

2 Some remarks on Galois theory of representations.

Let $F/k = \mathbb{Q}$ be a finite Galois extension, $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}'$ be a reductive Lie algebra over k , $\mathfrak{c} \subset \mathfrak{g}$ be its center and $\mathfrak{g}' \subset \mathfrak{g}$ be its derived subalgebra. Let $S = \text{Gal}(F/k)$ and $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ be a Galois-invariant (i.e. such that $g(\mathfrak{h}) = \mathfrak{h}$ for any $g \in S$) splitting Cartan subalgebra. Let B be a basis of the root system R of $(\mathfrak{g} \otimes_k F, \mathfrak{h})$. In what follows we assume that all the representations of \mathfrak{g} we are dealing with are finite-dimensional and can be integrated to representations of a reductive algebraic group with Lie algebra \mathfrak{g} (in order to guarantee their complete reducibility).

Let $\rho: \mathfrak{g} \rightarrow \text{End}_k(W)$ be a representation of \mathfrak{g} over k and $W \otimes_k F = \bigoplus_{\alpha} V_{\alpha}$ its decomposition into irreducible subrepresentations over F . Let $\rho_{\alpha} = \rho|_{V_{\alpha}}$ be an irreducible representation of $\mathfrak{g} \otimes_k F$ with primitive element $v_{\alpha} \in W \otimes_k F$ with highest weight $\omega_{\alpha} \in \text{Hom}_F(\mathfrak{h}, F)$ (with respect to B). Then for any $g \in S$, $\rho_{\alpha}^g := \rho|_{g(V_{\alpha})}$ is an irreducible representation of $\mathfrak{g} \otimes_k F$ with primitive element $g(v_{\alpha}) \in W \otimes_k F$ with highest weight $g \circ \omega_{\alpha} \circ g^{-1} \in \text{Hom}_F(\mathfrak{h}, F)$ with respect to the basis $g \circ B \circ g^{-1}$ of R . Since the Weyl group \mathcal{W}_R of R acts simply transitively on the set of bases of R , for any $g \in S$ there exists unique $w(g) \in \mathcal{W}_R$ such that $g \circ B \circ g^{-1} = w(g)(B)$. Hence ρ_{α}^g is an irreducible representation of $\mathfrak{g} \otimes_k F$ with primitive element $g(v_{\alpha}) \in W \otimes_k F$ with highest weight $\omega_{\alpha}^g := w(g)^{-1}(g \circ \omega_{\alpha} \circ g^{-1}) \in \text{Hom}_F(\mathfrak{h}, F)$ (with respect to B).

Lemma 3. *Suppose that $\rho_1: \mathfrak{g} \rightarrow \text{End}_k(W_1)$ and $\rho_2: \mathfrak{g} \rightarrow \text{End}_k(W_2)$ are two irreducible representations of \mathfrak{g} over k , $V_{\alpha} \subset W_1 \otimes_k F$ and $V_{\beta} \subset W_2 \otimes_k F$ are two irreducible subrepresentations of $\mathfrak{g} \otimes_k F$ over F . Then $W_1 \cong W_2$ as \mathfrak{g} -modules over k , if and only if there exist $\sigma, \tau \in S$ such that $(\rho_1|_{V_{\alpha}})^{\sigma} \cong (\rho_2|_{V_{\beta}})^{\tau}$ as $\mathfrak{g} \otimes_k F$ -modules over F .*

Proof: Schur's lemma. *QED*

Corollary. *If $\rho_\alpha: \mathfrak{g} \otimes_k F \rightarrow \text{End}_F(V_\alpha)$ is an irreducible representation of $\mathfrak{g} \otimes_k F$ over F , then there exists at most one irreducible representation $\rho: \mathfrak{g} \rightarrow \text{End}_k(W)$ of \mathfrak{g} over k , such that ρ_α is a subrepresentation of $\rho \otimes_k F$.*

Using the notation of the remark preceeding Lemma 3, let $W = \bigoplus_\gamma W_\gamma$ be a decomposition of ρ into irreducible subrepresentations over k . Then for any γ such that $V_\alpha \subset W_\gamma \otimes_k F$, by Galois descent we have:

$$\bigoplus_{\gamma': W_{\gamma'} \cong W_\gamma \text{ as } \mathfrak{g}\text{-modules}} W_{\gamma'} = \left(\bigoplus_{\alpha': \rho_\alpha^\tau \cong \rho_{\alpha'}^\sigma \text{ for some } \tau, \sigma \in S} V_{\alpha'} \right)^S.$$

Hence $\dim_k \left(\bigoplus_{\gamma': W_{\gamma'} \cong W_\gamma \text{ as } \mathfrak{g}\text{-modules}} W_{\gamma'} \right) = \dim_F \left(\bigoplus_{\alpha': \rho_\alpha^\tau \cong \rho_{\alpha'}^\sigma \text{ for some } \tau, \sigma \in S} V_{\alpha'} \right) = \sum_{\alpha': \exists \tau, \sigma \in S: \omega_\alpha^\tau = \omega_{\alpha'}^\sigma} \dim_F(V_{\alpha'}) = m_\alpha \cdot \dim_k(W_\gamma)$, where m_α is the multiplicity of W_γ in the decomposition above.

So, if $W_{\gamma_1}, \dots, W_{\gamma_p}$ are pairwise nonisomorphic (as \mathfrak{g} -modules) irreducible \mathfrak{g} -submodules of W over k (with the corresponding $\mathfrak{g} \otimes_k F$ -submodules $V_{\alpha_i} \subset W \otimes_k F$) appearing in the decomposition above, then $W = \bigoplus_i W_{\gamma_i}^{\oplus m_{\alpha_i}}$ and

$$\text{End}_{\mathfrak{g}}(W) \cong \prod_i \text{Mat}_{m_{\alpha_i} \times m_{\alpha_i}}(D_i) \text{ as } k\text{-algebras},$$

where $D_i = \text{End}_{\mathfrak{g}}(W_{\gamma_i})$, W_{γ_i} is the unique irreducible \mathfrak{g} -module over k such that $W_{\gamma_i} \otimes_k F$ contains V_{α_i} as a $\mathfrak{g} \otimes_k F$ -submodule over F and $m_{\alpha_i} = \left(\sum_{\alpha': \exists \sigma \in S: \omega_{\alpha'} = \omega_{\alpha_i}} \dim_F(V_{\alpha'}) \right) / \dim_k(W_{\gamma_i})$. We can also write:

$$m_{\alpha_i} = \frac{\dim_F(V_{\alpha_i}) \cdot \sum_{\sigma \in S} \text{mult}(\omega_{\alpha_i}^\sigma)}{n_{\omega_{\alpha_i}} \cdot \dim_k(W_{\gamma_i})},$$

where $\text{mult}(\omega)$ is the multiplicity of the irreducible representation of $\mathfrak{g} \otimes_k \mathbb{C}$ with highest weight ω (relative to the chosen \mathfrak{h} and B) in $W \otimes_k \mathbb{C}$ and n_ω is the stabilizer of ω under the action of the Galois group $S = \text{Gal}(F/k)$ on weights. Note that $\{\omega_{\alpha_i}\}$ is a set of representatives of the orbits of the action of S on the set of highest weights of irreducible representations of $\mathfrak{g} \otimes_k \mathbb{C}$ appearing as irreducible components of $W \otimes_k \mathbb{C}$.

This reduces the study of $\text{End}_{\mathfrak{g}}(W)$ to the study of the (uniquely determined) $(k = \mathbb{Q})$ -forms of irreducible $\mathfrak{g} \otimes_k \mathbb{C}$ -submodules of $W \otimes_k \mathbb{C}$ (i.e. $D_i = \text{End}_{\mathfrak{g}}(W_{\gamma_i})$ and $\dim_k(W_{\gamma_i})$) and the description of the Galois action (of the finite group $\text{Gal}(F/k)$) on the weights of $\mathfrak{g} \otimes_k \mathbb{C}$ over \mathbb{C} .

3 Description of the Galois action, Cartan subalgebras and bases of the root systems.

According to Section 2, we need to specify a splitting field F of \mathfrak{g} (which should be a Galois extension of k), a Galois-invariant splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ (i.e. \mathfrak{h} should be $\text{Gal}(F/k)$ -stable) and a basis B of the root system R of the split reductive Lie algebra $(\mathfrak{g} \otimes_k F, \mathfrak{h})$.

Let us assume that $\Phi = d_1 \cdot X_1^2 + \dots + d_m \cdot X_m^2$ (if $E = E_0 = L$ is totally real) or $\Phi = d_1 \cdot X_1 \bar{X}_1 + \dots + d_m \cdot X_m \bar{X}_m$ (if $E = E_0(\theta)$, $\theta^2 \in E_0 = L$ is a CM-field), where $d_i \in L$ for any i . In other words, we reduce the Hermitian (or quadratic) form Φ to a diagonal form, i.e. choose an orthogonal (with respect to Φ) basis of V such that X_i are the corresponding coordinates.

Let $k = \mathbb{Q}$ and F/k be a finite Galois extension such that F contains L , $\sqrt{d_i}$ for any i , $\sqrt{-1}$ and θ (if $E = E_0(\theta)$ is a CM-field, $\theta^2 \in E_0$).

Let $r = [L : k]$ and $\sigma_1, \dots, \sigma_r : L \hookrightarrow F$ be the list of all field embeddings of L into F .

3.1 Case of the totally real field.

Let us consider first the case $\mathfrak{g} = \text{Res}_{L/k}(\mathfrak{so}(\Phi)) \subset \mathfrak{so}(\phi)$ (i.e. $E = E_0$ is totally real). We will denote by $E_{i,j}$ a matrix with all entries equal to 0 except for the entry (i, j) which is equal to 1.

Let $\mathfrak{h}_0 = \text{Span}_L(A_1, \dots, A_l)$, where $l = \lfloor \frac{m}{2} \rfloor$ and $A_i = d_{m-i+1} \cdot E_{m-i+1,i} - d_i \cdot E_{i,m-i+1}$, $1 \leq i \leq l$. Let $\mathfrak{h}_i = \mathfrak{h}_0 \otimes_{L, \sigma_i} F \subset \mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$ and $\mathfrak{h} = \mathfrak{h}_1 \times \dots \times \mathfrak{h}_r \subset \bigoplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F) \cong \text{Res}_{L/k}(\mathfrak{so}(\Phi)) \otimes_k F = g \otimes_k F$. Then $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ is a splitting Cartan subalgebra.

Note that over F we have $\Phi = d_1 \cdot X_1^2 + \dots + d_m \cdot X_m^2 = \sum_{i=1}^l Y_i \cdot Y_{-i} + \epsilon Y_0^2$, where $\epsilon = 0$, if m is even, $\epsilon = 1$, if m is odd, $Y_i = \sqrt{d_i} \cdot X_i + \sqrt{-d_{m-i+1}} \cdot X_{m-i+1}$, $Y_{-i} = \sqrt{d_i} \cdot X_i - \sqrt{-d_{m-i+1}} \cdot X_{m-i+1}$ and $Y_0 = \sqrt{d_{l+1}} \cdot X_{l+1}$.

This implies that for any i, j we have $A_j \otimes_{L, \sigma_i} 1 = \Gamma_j \cdot H_j$, where $\Gamma_j = -\sqrt{\sigma_i(d_j)} \cdot \sqrt{-\sigma_i(d_{m-j+1})} \in F$ ($1 \leq j \leq l$) (in future we will be writing d_j instead of $\sigma_i(d_j)$) and $H_j = E_{j,j} - E_{-j,-j}$ (using notation from [1], §13). Hence for any i subalgebra $\mathfrak{h}_i \subset \mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$ is the same splitting Cartan subalgebra as in [1], §13. By construction $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ is Galois-invariant.

Let R_0 be the root system of type B_l , if $m = 2l + 1$ (respectively, of type D_l , if $m = 2l$) from [1], §13, i.e. $R_0 = \{\pm \epsilon_p, \pm \epsilon_p \pm \epsilon_q\}$ (respectively, $R_0 = \{\pm \epsilon_p \pm \epsilon_q\}$) with basis $B_0 = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{l-1} - \epsilon_l, \epsilon_l\}$ (respectively, $B_0 = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{l-1} - \epsilon_l, \epsilon_{l-1} + \epsilon_l\}$) (using notation from [1], §13).

Then for any i the root system of $(so(\Phi) \otimes_{L, \sigma_i} F, h_0 \otimes_{L, \sigma_i} F)$ is $R_i = \{\pm \epsilon_p \otimes_{L, \sigma_i} \Gamma_p, \pm \epsilon_p \otimes_{L, \sigma_i} \Gamma_p \pm \epsilon_q \otimes_{L, \sigma_i} \Gamma_q\}$ with basis

$$B_i = \{\epsilon_1 \otimes_{L, \sigma_i} \Gamma_1 - \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2, \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2 - \epsilon_3 \otimes_{L, \sigma_i} \Gamma_3, \dots, \epsilon_{l-1} \otimes_{L, \sigma_i} \Gamma_{l-1} - \epsilon_l \otimes_{L, \sigma_i} \Gamma_l, \epsilon_l \otimes_{L, \sigma_i} \Gamma_l\}$$

(respectively, $R_i = \{\pm \epsilon_p \otimes_{L, \sigma_i} \Gamma_p \pm \epsilon_q \otimes_{L, \sigma_i} \Gamma_q\}$ with basis

$$B_i = \{\epsilon_1 \otimes_{L, \sigma_i} \Gamma_1 - \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2, \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2 - \epsilon_3 \otimes_{L, \sigma_i} \Gamma_3, \dots, \epsilon_{l-1} \otimes_{L, \sigma_i} \Gamma_{l-1} - \epsilon_l \otimes_{L, \sigma_i} \Gamma_l, \epsilon_{l-1} \otimes_{L, \sigma_i} \Gamma_{l-1} + \epsilon_l \otimes_{L, \sigma_i} \Gamma_l\}.$$

Then $R = R_1 \sqcup \dots \sqcup R_r$ is the root system of $(\mathfrak{g} \otimes_k F, \mathfrak{h})$ and as a basis we can take $B = B_1 \sqcup \dots \sqcup B_r \subset R$.

The action of the Galois group $S = Gal(F/k)$ on weights reduces to its action by permutation on factors of $R_1 \times \dots \times R_r$ (or on the left cosets $Gal(F/k)/Gal(F/L)$) and to switching signs in front of various Γ_p .

Note the isomorphism of root systems $R \cong R_0 \sqcup \dots \sqcup R_0$ (r factors) under which basis B is identified with $B_0 \sqcup \dots \sqcup B_0$ (r factors).

Let $w_p \in \mathcal{W}_{R_0}$ (where \mathcal{W}_R denotes the Weyl group of a root system R) be the element of the Weyl group such that $w_p(B_0) = \sigma_p(B_0)$, where σ_p is a linear transformation of the \mathbb{Q} -vector space generated by the roots of R_0 which switches the sign in front of ϵ_p and does not change other ϵ_q 's. Then in the notation of Section 2 for any $g \in S$, $\omega_\alpha^g = (\prod_{p \in P_1(g)} w_p^{-1}) \sqcup \dots \sqcup (\prod_{p \in P_r(g)} w_p^{-1})(g \circ \omega_\alpha \circ g^{-1}) \in Hom_F(\mathfrak{h}, F)$, where $P_i(g) = \{p \mid g^{-1}(\epsilon_p \otimes_{L, \sigma_i} \Gamma_p) = -\epsilon_p \otimes_{L, g^{-1} \circ \sigma_i} \Gamma_p\}$.

3.2 Case of the CM-field.

Now let us consider the case $\mathfrak{g} = Res_{L/k}(\mathfrak{u}(\Phi)) \subset \mathfrak{so}(\phi)$ (i.e. $E = E_0(\theta)$ is a CM-field, $\theta^2 \in E_0 = L$).

Let $\mathfrak{h}_0 = Span_L(A_1, \dots, A_m)$, where $A_i = \theta \cdot E_{i,i}$, $\mathfrak{h}_i = \mathfrak{h}_0 \otimes_{L, \sigma_i} F \subset \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)$ and $\mathfrak{h} = \mathfrak{h}_1 \times \dots \times \mathfrak{h}_r \subset \oplus_{i=1}^r (\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F) \cong \mathfrak{gl}(m, F)^{\otimes r} \cong Res_{L/k}(\mathfrak{u}(\Phi)) \otimes_k F = \mathfrak{g} \otimes_k F$. Then $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ is a splitting Cartan subalgebra.

Note that over F we have $\Phi = d_1 \cdot X_1 \bar{X}_1 + \dots + d_m \cdot X_m \bar{X}_m = Y_1 \bar{Y}_1 + \dots + Y_m \bar{Y}_m$, where $Y_i = \sqrt{d_i} \cdot X_i$. Hence for any i, j we have $A_j \otimes_{L, \sigma_i} 1 = \theta \cdot E_{j,j}$ (more precisely we have to write $\sqrt{\sigma_i(\theta^2)}$ instead of θ here) in $\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)$ and so for any i subalgebra $\mathfrak{h}_i \subset \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)$ is the same splitting Cartan subalgebra as in [1], §13. By construction $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ is Galois-invariant.

Let R_0 be the root system of type A_{m-1} (for the reductive Lie algebra $\mathfrak{gl}(m) = \mathfrak{c} \oplus \mathfrak{sl}(m)$, where $\mathfrak{c} \subset \mathfrak{gl}(m)$ is the center), i.e. $R_0 = \{\epsilon_p - \epsilon_q\}_{p \neq q}$ with basis $B_0 = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m\}$.

Then for any i the root system of $(\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F, \mathfrak{h}_i) \cong (\mathfrak{gl}(m), \text{diagonal matrices})$ is $R_i = \{\epsilon_p \otimes_{L, \sigma_i} \theta - \epsilon_q \otimes_{L, \sigma_i} \theta\}$ with basis $B_i = \{\epsilon_1 \otimes_{L, \sigma_i} \theta - \epsilon_2 \otimes_{L, \sigma_i} \theta, \dots, \epsilon_{m-1} \otimes_{L, \sigma_i} \theta - \epsilon_m \otimes_{L, \sigma_i} \theta\}$. Then $R = R_1 \sqcup \dots \sqcup R_r$ is the root system of $(\mathfrak{g} \otimes_k F, \mathfrak{h})$ and as a basis we can take $B = B_1 \sqcup \dots \sqcup B_r \subset R$.

The action of the Galois group $S = \text{Gal}(F/k)$ on weights reduces to its action by permutation on factors of $R_1 \sqcup \dots \sqcup R_r$ (or on the left cosets $\text{Gal}(F/k)/\text{Gal}(F/L)$) and to multiplication of various θ by -1 .

Note the isomorphism of root systems $R \cong R_0 \sqcup \dots \sqcup R_0$ (r factors) under which basis B is identified with $B_0 \sqcup \dots \sqcup B_0$ (r factors).

Let $w_0 \in \mathcal{W}_{R_0}$ be such that $w_0(B_0) = -B_0$. Then in the notation of Section 2 for any $g \in S$, $\omega_\alpha^g = (w_0)^{-P_1(g)} \sqcup \dots \sqcup (w_0)^{-P_r(g)}(g \circ \omega_\alpha \circ g^{-1}) \in \text{Hom}_F(\mathfrak{h}, F)$, where $P_i(g) = 1$, if $g^{-1}(\epsilon_p \otimes_{L, \sigma_i} \theta) = -\epsilon_p \otimes_{L, g^{-1} \circ \sigma_i} \theta$ and $P_i(g) = 0$ otherwise (the action of w_0 is extended to the center of $\mathfrak{gl}(m, F)$ as multiplication by -1).

4 Decomposition of the restriction of the spin representation over a splitting field.

In order to apply the general statements of Section 2, we need to decompose the F -linear extension of the restriction of the spin representation of $\mathfrak{so}(\phi)$ in $C^+(V)$ to $\mathfrak{g} \subset \mathfrak{so}(\phi)$ over F . For this we need to describe the embedding of Cartan subalgebras induced by the embedding of Lie algebras $\mathfrak{g} \otimes_k F \subset \mathfrak{so}(\phi) \otimes_k F$.

Lemma 2. *If E is totally real, then the Lie algebra homomorphism $\oplus_{i=1}^r \mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F \subset \mathfrak{so}(\oplus_{i=1}^r (\Phi \otimes_{L, \sigma_i} F)) = \mathfrak{so}(\phi) \otimes_k F$ sends (M_1, \dots, M_r) to $\text{diag}(M_1, \dots, M_r)$.*

If $E = E_0(\theta)$ is a CM-field (and $\theta^2 \in E_0 = L$ as usual), then the Lie algebra homomorphism $\oplus_{i=1}^r \mathfrak{gl}(m, F) \cong \oplus_{i=1}^r \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \subset \mathfrak{so}(\oplus_{i=1}^r ((\Phi \otimes_{E, \sigma_i} F) \oplus (\Phi \otimes_{E, \bar{\sigma}_i} F))) = \mathfrak{so}(\phi) \otimes_k F$ (where in the last formula σ_i and $\bar{\sigma}_i$ denote the two extensions of σ_i to an embedding of E/k into F/k) sends (M_1, \dots, M_r) to $\text{diag}(M_1, -\Phi \cdot M_1^T \cdot \Phi^{-1}, \dots, M_r, -\Phi \cdot M_r^T \cdot \Phi^{-1})$.

Proof: One should notice that $\text{Res}_{L/k}$ on vector spaces over L is the forgetful functor to the vector spaces over k . Hence on the $\text{Res}_{L/k}(\mathfrak{so}(\Phi))$ (respectively, $\text{Res}_{L/k}(\mathfrak{u}(\Phi))$), which is the Galois-invariant subspace of the source, our homomorphisms have exactly the form needed. Extending scalars to F gives the result. See also Proposition 3.8 in [13] and [14]. *QED*

4.1 Case of the totally real field.

Let $E = L$ be a totally real field.

For any $i = 1, \dots, r, j = 1, \dots, l$ (where $l = \lfloor \frac{m}{2} \rfloor$) let $\hat{H}_j^i = \sigma_i(d_{m-j+1}) \cdot E_{m-j+1+m(i-1), j+m(i-1)} - \sigma_i(d_j) \cdot E_{j+m(i-1), m-j+1+m(i-1)} \in \mathfrak{so}(\phi) \otimes_k F$. Then \hat{H}_j^i are linearly independent elements of the splitting Cartan subalgebra $\hat{\mathfrak{h}} \subset \mathfrak{so}(\phi) \otimes_k F$ described in [1], §13. They form a basis of $\hat{\mathfrak{h}}$, if m is even or $r = 1$. If m is odd and $r \geq 2$, then \hat{H}_j^i together with $\hat{H}_{l+1}^1, \dots, \hat{H}_{l+1}^{\lfloor \frac{r}{2} \rfloor}$ form a basis of $\hat{\mathfrak{h}}$, if we take $\hat{H}_{l+1}^i = \sigma_{r-i+1}(d_{l+1}) \cdot E_{(l+1)(r-i+1), (l+1)i} - \sigma_i(d_{l+1}) \cdot E_{(l+1)i, (l+1)(r-i+1)}$, $1 \leq i \leq \lfloor \frac{r}{2} \rfloor$.

Let us denote by $\{\hat{\epsilon}_j^i\}$ the corresponding dual basis of $\hat{\mathfrak{h}}^* = \text{Hom}_F(\hat{\mathfrak{h}}, F)$. Its elements differ from the elements of the corresponding basis of the dual Cartan subalgebra considered in [1], §13 by scalar factors of the form $-\sqrt{\sigma_i(d_j)} \cdot \sqrt{-\sigma_i(d_{m-j+1})}$.

Lemma 2 above implies that the restriction of $\hat{\epsilon}_{l+1}^i$ to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ is zero, while for any $j \leq l$ the restriction of $\hat{\epsilon}_j^i$ to \mathfrak{h} is the corresponding element of the dual basis of \mathfrak{h}^* of the basis $\{A_j \otimes_{L, \sigma_i} 1 \mid 1 \leq i \leq r, 1 \leq j \leq l\}$ of \mathfrak{h} .

If $m \cdot r = \dim_k(V) \geq 5$, then according to [1], §13 the weights of the spin representation of $\mathfrak{so}(\phi) \otimes_k F$ in $C^+(V) \otimes_k F$ (V is considered as a vector space over k) are $\frac{1}{2} \sum_{i,j} \hat{\epsilon}_j^i - \sum_{(i,j) \in I} \hat{\epsilon}_j^i$, where I runs over the subsets of the set of parameters i and j (i.e. $I \subset \{(i,j) \mid 1 \leq j \leq l \text{ and } 1 \leq i \leq r \text{ or (if } m \text{ is odd and } r \geq 2) j = l+1 \text{ and } 1 \leq i \leq \lfloor \frac{r}{2} \rfloor\})$) and each weight has multiplicity $\frac{\dim_k(C^+(V))}{2^{\lfloor mr/2 \rfloor}} = 2^{mr-1-\lfloor \frac{mr}{2} \rfloor}$.

As it was remarked in [14], Lemma 5.5, this implies (if $m \geq 5$) that the restrictions of these weights to $\mathfrak{h} \subset \hat{\mathfrak{h}}$ are exactly the weights of the exterior tensor product of the spin representations of $\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$ in $C^+(V) \otimes_{L, \sigma_i} F$ (V is considered as a vector space over L), $1 \leq i \leq r$, taken with multiplicity $\frac{2^{mr-1-\lfloor mr/2 \rfloor}}{(2^{m-1-l})^r} = 2^{r-1}$, if m is even, or with multiplicity $\frac{2^{mr-1-\lfloor mr/2 \rfloor}}{(2^{m-1-l})^r} \cdot 2^{\lfloor \frac{r}{2} \rfloor} = 2^{r-1}$, if m is odd.

Corollary 1. *If $E = E_0 = L$ is totally real, then the restriction of the spin representation $\rho: \mathfrak{so}(\phi) \otimes_k F \rightarrow \text{End}_F(C^+(V \otimes_k F))$ to $\mathfrak{g} \otimes_k F = \bigoplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F) \subset \mathfrak{so}(\phi) \otimes_k F$ is the exterior tensor product $\Gamma \cdot (\rho_1 \boxtimes \dots \boxtimes \rho_r)$ of spin representations $\rho_i: \mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F \rightarrow \text{End}_F(C^+(V \otimes_{L, \sigma_i} F))$ with multiplicity $\Gamma = 2^{r-1}$.*

4.2 Case of the CM-field.

Let $E = E_0(\theta), \theta^2 \in E_0 = L$ be a CM-field.

For any $i = 1, \dots, r, j = 1, \dots, m$ let $\hat{H}_j^i = E_{j+2m(i-1), j+2m(i-1)} - E_{j+m+2m(i-1), j+m+2m(i-1)} \in \mathfrak{so}(\phi) \otimes_k F$. Then \hat{H}_j^i form a basis of the splitting Cartan subalgebra $\hat{\mathfrak{h}} \subset \mathfrak{so}(\phi) \otimes_k F$ described in [1], §13. Let us denote by $\{\hat{\epsilon}_j^i\}$ the corresponding dual basis of $\hat{\mathfrak{h}}^* = \text{Hom}_F(\hat{\mathfrak{h}}, F)$.

This is the same Cartan subalgebra and the same basis as considered in [1], §13.

Lemma 2 above implies that the restriction of $\hat{\epsilon}_j^i$ to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} \otimes_k F$ is the element $(0, \dots, \hat{\epsilon}_j, \dots, 0)$ (with 0 outside of the i -th spot) of the Cartan subalgebra (consisting of diagonal matrices) of $\mathfrak{gl}(m, F)^{\oplus r}$, where $\hat{\epsilon}_j \cong E_{j,j} \in \mathfrak{gl}(m, F)$ is the j -th element of the dual basis of the Cartan subalgebra of $\mathfrak{gl}(m, F)$ considered in [1], §13.

If $m \cdot r = \frac{1}{2} \cdot \dim_k(V) \geq 3$, then according to [1], §13 the weights of the spin representation of $\mathfrak{so}(\phi) \otimes_k F$ in $C^+(V) \otimes_k F$ (V is considered as a vector space over k) are $\frac{1}{2} \sum_{i,j} \hat{\epsilon}_j^i - \sum_{(i,j) \in I} \hat{\epsilon}_j^i$. Here I runs over the subsets of $[1, \dots, r] \times [1, \dots, m]$. Each weight has multiplicity $\frac{\dim_k(C^+(V))}{2^{mr}} = 2^{mr-1}$ ([1], §13).

Suppose $m \geq 2$. Then the restrictions of these weights to $\mathfrak{h} \subset \hat{\mathfrak{h}}$ are exactly the weights of the exterior tensor product of the exterior algebra representations of $\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)$ in $\wedge_E^*(V) \otimes_{E, \sigma_i} F$ (V is considered as a vector space over E) twisted by $D^{-1/2}$, $1 \leq i \leq r$. Here D^c , $c \in \mathbb{Q}$ denotes the representation of $\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F) = \mathfrak{c} \oplus \mathfrak{sl}(m, F)$ in $\wedge_E^m(V) \otimes_{E, \sigma_i} F \cong \wedge_F^m(V \otimes_{E, \sigma_i} F)$ such that $\mathfrak{sl}(m, F)$ acts trivially, while $1 \in F \cong \mathfrak{c}$ acts as $c \cdot Id$. In other words, $D^c: \mathfrak{gl}(m, F) \rightarrow \text{End}_F(\wedge_E^m(V) \otimes_{E, \sigma_i} F)$, $M \mapsto c \cdot \text{Tr}(M) \cdot Id$.

Indeed, for any i , $\sum_j \hat{\epsilon}_j^i$ restricts to 0 to the Cartan subalgebra of the semi-simple part $\mathfrak{sl}(m, F) \subset \mathfrak{gl}(m, F) \cong \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F$ and to $m \cdot Id_F$ to the center $F \cong \mathfrak{c} \subset \mathfrak{gl}(m, F) \cong \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F$.

The exterior tensor product above has multiplicity $\Gamma = 2^{mr-1}$. Indeed, $\dim_F(C^+(V) \otimes_k F) = 2^{2mr-1}$ and $\dim_F(\wedge_E^*(V) \otimes_{E, \sigma_i} F) = 2^m$. Hence the dimension of the exterior tensor product is $(\dim_F(\wedge_E^*(V) \otimes_{E, \sigma_i} F))^r = 2^{mr}$ and so the multiplicity is $2^{2mr-1}/2^{mr} = 2^{mr-1}$.

Corollary 2. *If $E = E_0(\theta)$, $\theta^2 \in E_0 = L$ is a CM-field, then the restriction of the spin representation $\rho: \mathfrak{so}(\phi) \otimes_k F \rightarrow \text{End}_F(C^+(V \otimes_k F))$ to $\mathfrak{g} \otimes_k F = \bigoplus_{i=1}^r (\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F) \cong \mathfrak{gl}(m, F)^{\oplus r} \subset \mathfrak{so}(\phi) \otimes_k F$ is the exterior tensor product $\Gamma \cdot (\rho_1 \boxtimes \dots \boxtimes \rho_r)$ of exterior algebra representations $\rho_i: \mathfrak{gl}(m, F) \rightarrow \text{End}_F(\wedge_F^*(V \otimes_{E, \sigma_i} F) \otimes_F F)$ twisted by one-dimensional representations $D^{-1/2}: \mathfrak{gl}(m, F) \rightarrow \text{End}_F(F) \cong F$, $M \mapsto (-\frac{1}{2m}) \cdot \text{Tr}(M)$ with multiplicity $\Gamma = 2^{mr-1}$.*

Remark. ρ_i is a double-valued 'spin' representation of $GL(m, F)$.

From these Corollaries one can deduce the highest weights of irreducible subrepresentations over F of the restriction to $\mathfrak{g} \otimes_k F \subset \mathfrak{so}(\phi) \otimes_k F$ of the spin representation $\rho: \mathfrak{so}(\phi) \rightarrow \text{End}_k(C^+(V))$. Then one can use the description of the Galois action of $S = \text{Gal}(F/k)$ on weights of $\mathfrak{g} \otimes_k F$ given above in order to break down the highest weights into orbits $\{S \cdot \omega_1, \dots, S \cdot \omega_t\}$. Let us denote the dimension of the irreducible representation of $\mathfrak{g} \otimes_k F$ with highest weight ω_i by d_i . Let $\hat{\rho}_i: \mathfrak{g} \rightarrow \text{End}_k(W_i)$ be the (unique) irreducible representation of \mathfrak{g} over k such that $W_i \otimes_k F$ contains the irreducible representation of $\mathfrak{g} \otimes_k F$ with highest weight ω_i as a $(\mathfrak{g} \otimes_k F)$ -submodule. Then our analysis in Section 2 implies:

Theorem 1.

$$\text{End}(KS(X))_{\mathbb{Q}} \cong \text{End}_{\mathfrak{g}}(W) \cong \prod_i \text{Mat}_{m_i \times m_i}(D_i) \text{ as } \mathbb{Q} - \text{algebras},$$

where $D_i = \text{End}_{\mathfrak{g}}(W_i)$, $m_i = (d_i/\dim_k(W_i)) \cdot \sum_{\omega \in S \cdot \omega_i} \text{mult}(\omega)$ and $\text{mult}(\omega)$ is the multiplicity of the irreducible subrepresentation of the representation of $\mathfrak{g} \otimes_k F$ on $C^+(V \otimes_k F)$ with highest weight ω .

Remark. In the analysis above we assumed that $m = \dim_E V \geq 5$ (if E is totally real) or $m \geq 2$ (if E is a CM-field and $r = [E : k]/2 \geq 2$) or $m \geq 3$ (if E is a CM-field and $r = [E : k]/2 = 1$). In the case of small m Lie algebras we consider 'degenerate' and require a separate consideration.

5 \mathbb{Q} -forms of spin representations.

Let us describe more explicitly \mathbb{Q} -forms W_i above or at least the corresponding primary representations. We will use corestriction of algebraic structures, as in [14], §6 and (in the case of totally real fields) representation spaces which we are going to construct in the following subsection.

5.1 Galois-invariant sums of ideals of Clifford algebra.

Let $k = \mathbb{Q}$, $E = L$ be a totally real number field, $r = [L : k]$. Let $\Phi = d_1 \cdot X_1^2 + \dots + d_m \cdot X_m^2$ with respect to basis $\{e_1, \dots, e_m\}$ of V , $m = \dim_L V$. Let F/k be a finite Galois extension containing L , $\sqrt{-1}$ and $\sqrt{d_i}$ for all i . Let $\sigma_1, \dots, \sigma_r : L \hookrightarrow F$ be all the field embeddings over k .

Let $f_i = \frac{1}{\sqrt{d_i}} \cdot e_i + \frac{1}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1}$, $f_{-i} = \frac{1}{\sqrt{d_i}} \cdot e_i - \frac{1}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1}$, $1 \leq i \leq l = \lfloor \frac{m}{2} \rfloor$ and $f_0 = \frac{1}{\sqrt{d_{l+1}}} \cdot e_{l+1}$. Then $\{f_i, f_{-i} \mid 1 \leq i \leq l\}$ (if m is even) or $\{f_0, f_i, f_{-i} \mid 1 \leq i \leq l\}$ (if m is odd) is a basis of $V \otimes_{L, \sigma_i} F$, where we denote $\sigma_i(d_j)$ by d_j . With respect to this basis $\Phi = 2 \sum_{i=1}^l Y_i \cdot Y_{-i} + \epsilon Y_0^2$, where $\epsilon = (1 - (-1)^m)/2$.

5.1.1 Even dimension.

Assume that m is even. Let $f_{\alpha_1, \dots, \alpha_l}^i = f_{\alpha_1 \cdot 1} \cdot \dots \cdot f_{\alpha_l \cdot l} \in C(V \otimes_{L, \sigma_i} F)$ for various $\alpha_i \in \{\pm 1\}$ and $I_{\alpha_1, \dots, \alpha_l}^i = C(V \otimes_{L, \sigma_i} F) \cdot f_{\alpha_1, \dots, \alpha_l}^i$, $1 \leq i \leq r$. $I_{\alpha_1, \dots, \alpha_l}^i$ are left ideals of the Clifford algebra $C(V \otimes_{L, \sigma_i} F)$ viewed as F -vector subspaces.

Consider the direct sum of F -vector spaces

$$\tilde{C}(V \otimes_{L, \sigma_i} F) = \tilde{C}(V) \otimes_{L, \sigma_i} F = \bigoplus_{\alpha_1, \dots, \alpha_l \in \{\pm 1\}} I_{\alpha_1, \dots, \alpha_l}^i.$$

Note that $g(f_i) \in \{\pm f_i, \pm f_{-i}\}$ for any i and $g \in S$. Hence the Galois group $S = \text{Gal}(F/k)$ acts on $\tilde{C}(V \otimes_{L, \sigma_i} F)$ (by sending an element of the summand $I_{\alpha_1, \dots, \alpha_l}$ to its image under the action of S on $C(V \otimes_{L, \sigma} F)$ viewed as an element of the summand $I_{\beta_1, \dots, \beta_l}$, where $f_{\beta_1, \dots, \beta_l}$ is upto a scalar factor the image of $f_{\alpha_1, \dots, \alpha_l}$).

It follows from the construction that F -vector subspaces $\bigoplus_{i=1}^r I_{\alpha_1^i, \dots, \alpha_l^i}^i \subset \bigoplus_{i=1}^r \tilde{C}(V) \otimes_{L, \sigma_i} F$ for various choices of $\alpha_j^i \in \{\pm 1\}$ are permuted among themselves under the action of the Galois group $S = \text{Gal}(F/k)$.

Remark. For any $\alpha_1, \dots, \alpha_l$ the left ideal $I_{\alpha_1, \dots, \alpha_l}^i \subset C(V \otimes_{L, \sigma_i} F)$ is an $(\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F)$ -subrepresentation of the spin representation, which is either irreducible (if m is odd) or is the sum of two irreducible and non-isomorphic (semi-spin) representations [2], [4]. In the latter case, let us write $I_{\alpha_1, \dots, \alpha_l}^i = I_{\alpha_1, \dots, \alpha_l}^{i,+} \oplus I_{\alpha_1, \dots, \alpha_l}^{i,-}$ for the corresponding (unique) decomposition.

5.1.2 Odd dimension.

Assume that m is odd. Let $f_{\alpha_1, \dots, \alpha_l, \gamma}^i = f_{\alpha_1 \cdot 1} \cdot \dots \cdot f_{\alpha_l \cdot l} \cdot (1 + \gamma \cdot f_0) \in C(V \otimes_{L, \sigma_i} F)$ for various $\alpha_i, \gamma \in \{\pm 1\}$ and $I_{\alpha_1, \dots, \alpha_l, \gamma}^i = C(V \otimes_{L, \sigma_i} F) \cdot f_{\alpha_1, \dots, \alpha_l, \gamma}^i$, $1 \leq i \leq r$. $I_{\alpha_1, \dots, \alpha_l, \gamma}^i$ are left ideals of the Clifford algebra $C(V \otimes_{L, \sigma_i} F)$ viewed as F -vector subspaces.

Consider the direct sum of F -vector spaces

$$\tilde{C}(V \otimes_{L, \sigma_i} F) = \tilde{C}(V) \otimes_{L, \sigma_i} F = \bigoplus_{\alpha_1, \dots, \alpha_l, \gamma \in \{\pm 1\}} I_{\alpha_1, \dots, \alpha_l, \gamma}^i.$$

Note that $g(1 + \gamma \cdot f_0) = (1 \pm \gamma \cdot f_0)$ for any $g \in S$. Hence the Galois group $S = \text{Gal}(F/k)$ acts on $\tilde{C}(V \otimes_{L, \sigma_i} F)$ (by sending an element of the summand $I_{\alpha_1, \dots, \alpha_l, \gamma}$ to its image under the action of S on $C(V \otimes_{L, \sigma} F)$ viewed as an element of the summand $I_{\beta_1, \dots, \beta_l, \gamma'}$, where $f_{\beta_1, \dots, \beta_l, \gamma'}$ is upto a scalar factor the image of $f_{\alpha_1, \dots, \alpha_l, \gamma}$).

It follows from the construction that F -vector subspaces $\bigoplus_{i=1}^r I_{\alpha_1^i, \dots, \alpha_l^i, \gamma^i}^i \subset \bigoplus_{i=1}^r \tilde{C}(V) \otimes_{L, \sigma_i} F$ for various choices of $\alpha_j^i, \gamma^i \in \{\pm 1\}$ are permuted among themselves under the action of the Galois group $S = \text{Gal}(F/k)$.

Remark. For any $\alpha_1, \dots, \alpha_l, \gamma$ the left ideal $I_{\alpha_1, \dots, \alpha_l, \gamma}^i \subset C(V \otimes_{L, \sigma_i} F)$ is an irreducible $(\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F)$ -subrepresentation of the spin representation (since m is odd by assumption) [2], [4].

We will use $\tilde{C}(V \otimes_{L, \sigma_i} F)$ as representation spaces of $(\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F)$ (the direct sum of its representations on the left ideals of the Clifford algebra) in order to construct primary \mathbb{Q} -forms of spin representations.

5.2 Case of the totally real field and odd dimension.

Let $E = E_0 = L$ be totally real and $m = \dim_L V$ odd. Let $\Sigma_i \subset C^+(V \otimes_{L, \sigma_i} F)$, $1 \leq i \leq r$ be the irreducible subrepresentation of the spin representation of $\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$. Then $\Sigma_1 \otimes_F \dots \otimes_F \Sigma_r$ is an irreducible representation of $\oplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F) = \mathfrak{g} \otimes_k F$.

Let $\tilde{C}(V \otimes_{L, \sigma_i} F) = \oplus_p S_p^i$ be a decomposition into irreducible components of the representation of $\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$ considered above. Let Ω' be the finite set of F -vector subspaces of $\tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$ (or of $C(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F C(V \otimes_{L, \sigma_r} F)$) of the form $S_{p_1}^1 \otimes_F \dots \otimes_F S_{p_r}^r$ for various p_1, \dots, p_r . These subspaces are irreducible subrepresentations of the exterior tensor product of spin representations as a representation of $\oplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F)$.

Galois group $S = \text{Gal}(F/k)$ acts on Ω' . Take any element $S_{p_1}^1 \otimes_F \dots \otimes_F S_{p_r}^r$ of Ω' . Let $U \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$ be the sum of the elements of Ω' (as subspaces of $\tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$) lying in the S -orbit of $S_{p_1}^1 \otimes_F \dots \otimes_F S_{p_r}^r$. Then $U \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$ is an S -submodule.

Since the actions of $\mathfrak{g} \subset \mathfrak{g} \otimes_k F$ and $S = \text{Gal}(F/k)$ commute, by Galois descent

$$(U)^S \cong ((\Sigma_1 \otimes_F \dots \otimes_F \Sigma_r)^{\oplus n_0})^S$$

is a primary representation of \mathfrak{g} over k of dimension $n_0 \cdot 2^{l \cdot r}$, which contains $\Sigma_1 \otimes_F \dots \otimes_F \Sigma_r$ after extending scalars to F .

Multiplicity n_0 is the length of the S -orbit in Ω' of the chosen element $S_{p_1}^1 \otimes_F \dots \otimes_F S_{p_r}^r$ of Ω' .

Remark. We will use notation introduced above. Consider the action of $S = \text{Gal}(F/k)$ on 2^{l+1} elements (or more precisely on the lines generated by them) $f_{\beta_1, \dots, \beta_l, \gamma}$ of $C(V \otimes_L F)$ for various $\beta_1, \dots, \beta_l, \gamma$ by sign changes in front of $\sqrt{d_i}$'s and $\sqrt{-d_{m-i+1}}$'s in the definition of f_i in terms of e_j (see notation above). Then (if we choose all S_{p_i} to be the same)

$$n_0 = \frac{\text{order of } S = \text{Gal}(F/k)}{\text{order of the stabilizer of } f_{1, \dots, 1, 1}}.$$

5.3 Case of the totally real field and even dimension.

Let $E = E_0 = L$ be a totally real field and $m = \dim_L V$ even. Let $\Sigma_i^+, \Sigma_i^- \subset C^+(V \otimes_{L, \sigma_i} F)$, $1 \leq i \leq r$ be irreducible (semi-spin) subrepresentations of the spin representation of $\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$.

Consider the finite set Ω of F -vector spaces of the form $\Sigma_1^{\alpha_1} \otimes_F \dots \otimes_F \Sigma_r^{\alpha_r}$ for various $\alpha_i \in \{+, -\}$. They are exactly the irreducible components of the exterior tensor product of spin representations $\Sigma_i = \Sigma_i^+ \oplus \Sigma_i^- \subset C^+(V \otimes_{L, \sigma_i} F)$ of $\oplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F)$ (see [1], §13, [2], [4]). They are also the isomorphism classes of simple $\oplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F)$ -submodules of $C(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F C(V \otimes_{L, \sigma_r} F)$. Let $\tilde{C}(V \otimes_{L, \sigma_i} F) = \oplus_p S_p^i$ be a decomposition

into irreducible components of the representation of $\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$ considered above. Let Ω' be the finite set of F -vector subspaces of $C(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F C(V \otimes_{L, \sigma_r} F)$ (or of $\tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$) of the form $S_{p_1}^1 \otimes_F \dots \otimes_F S_{p_r}^r$ for various p_1, \dots, p_r . These subspaces are irreducible subrepresentations of the exterior tensor product of spin representations as a representation of $\oplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F)$.

Galois group $S = \text{Gal}(F/k)$ acts naturally on both Ω and Ω' . Let $\Omega_1, \dots, \Omega_u$ be the orbits of S on Ω . For any i choose $(\alpha_1, \dots, \alpha_r) \in \Omega_i$ and define $U_i \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$ to be the sum of the elements of Ω' (as subspaces of $\tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$) lying in the S -orbit of any $S_{p_1}^1 \otimes_F \dots \otimes_F S_{p_r}^r$, which is isomorphic to $\Sigma_1^{\alpha_1} \otimes_F \dots \otimes_F \Sigma_r^{\alpha_r}$ as an $\oplus_{i=1}^r (\mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F)$ -module.

Then $U_i \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$ is an S -submodule and

$$(U_i)^S \cong \left(\bigoplus_{(\alpha_1, \dots, \alpha_r) \in \Omega_i} (\Sigma_1^{\alpha_1} \otimes_F \dots \otimes_F \Sigma_r^{\alpha_r})^{\oplus n_{\alpha_1, \dots, \alpha_r}} \right)^S$$

is a primary representation of \mathfrak{g} over k of dimension $\sum_{(\alpha_1, \dots, \alpha_r) \in \Omega_i} n_{\alpha_1, \dots, \alpha_r} \cdot 2^{r(l-1)}$. These representations $(U_i)^S$, $1 \leq i \leq u$ contain all representations of $\mathfrak{g} \otimes_k F$ of the form $\Sigma_1^{\alpha_1} \otimes_F \dots \otimes_F \Sigma_r^{\alpha_r}$ after extending scalars to F .

Multiplicities $n_{\alpha_1, \dots, \alpha_r}$ can be computed as follows:

$$n_{\alpha_1, \dots, \alpha_r} = \frac{\text{order of the stabilizer of } (\alpha_1, \dots, \alpha_r) \in \Omega}{\text{order of the stabilizer of } (p_1, \dots, p_r) \in \Omega'}.$$

Remark. We will use notation introduced above. Consider the action of $S = \text{Gal}(F/k)$ on 2^l elements (or more precisely on the lines generated by them) $f_{\beta_1, \dots, \beta_l}$ of $C(V \otimes_L F)$ for various β_1, \dots, β_l by sign changes in front of $\sqrt{d_i}$'s and $\sqrt{-d_{m-i+1}}$'s in the definition of f_i in terms of e_j (see notation above). Then (if we choose all S_{p_i} to be the same)

$$(\text{stabilizer of } (p_1, \dots, p_r) \in \Omega') = (\text{stabilizer of } (\alpha_1, \dots, \alpha_r) \in \Omega) \cap (\text{stabilizer of } f_{1, \dots, 1}).$$

Remark. Instead of $\tilde{C}(V \otimes_L F)$ one can also consider the Clifford algebra $C(V \otimes_L F)$ (or its even part $C^+(V \otimes_L F)$). Then the corestriction of $C(V)$ (or of $C^+(V)$) (with V viewed as a vector space over L) from L to $k = \mathbb{Q}$ (or Galois-fixed subspaces of sums (inside of tensor products of $C(V) \otimes_L F$) of tensor products of $(\mathfrak{g} \otimes_k F)$ -invariant F -vector subspaces (or ideals used above) of $C(V) \otimes_L F$, which form a single Galois orbit) would be a representation of \mathfrak{g} over $\mathbb{Q} = k$, whose extension of scalars to F contains all the irreducible representations (and only them) of $\mathfrak{g} \otimes_k F$ over F which we need. In particular, in the case of odd m it would be another primary representation of \mathfrak{g} over k .

5.4 Case of the CM-field.

Let $E = E_0(\theta)$, $\theta^2 \in E_0 = L$ be a CM-field.

Note that the tautological representation of $\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F$ in $V \otimes_{L, \sigma_i} F$ splits into the direct sum of two representations of $\mathfrak{gl}(m, F) \cong \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F$:

$$V \otimes_{L, \sigma_i} F = (V \otimes_{E, \sigma_i} F) \oplus (V \otimes_{E, \bar{\sigma}_i} F),$$

where σ_i and $\bar{\sigma}_i$ are the two extensions of $\sigma_i: E_0 \rightarrow F$ to embeddings $E \rightarrow F$.

Since the exterior power representations $\wedge_F^p(V \otimes_{E, \bar{\sigma}_i} F)$ and $\wedge_F^p(V \otimes_{E, \sigma_i} F)$ of $\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)$ are identified by the Lie algebra automorphism $\mathfrak{gl}(m, F) \rightarrow \mathfrak{gl}(m, F)$, $M \mapsto -\Phi \cdot M^T \cdot \Phi^{-1}$, we have isomorphisms

$$\wedge_F^p(V \otimes_{E, \bar{\sigma}_i} F) \rightarrow \wedge_F^{m-p}(V \otimes_{E, \sigma_i} F) \otimes_F D^{-1}$$

and hence also isomorphisms

$$\tau_p: \wedge_F^p(V \otimes_{E, \bar{\sigma}_i} F) \otimes_F (E \otimes_{E, \bar{\sigma}_i} F) \rightarrow \wedge_F^{m-p}(V \otimes_{E, \sigma_i} F) \otimes_F D^{-1/2}, \quad 1 \leq p \leq m$$

of representations of $\mathfrak{gl}(m, F) \cong \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F$.

Let $\wedge_i^j \subset \wedge_F^*(V \otimes_{E, \sigma_i} F) \otimes_F F$, $1 \leq i \leq r$, $1 \leq j \leq m$ be the irreducible representation of $\mathfrak{gl}(m, F)$ on the F -vector space $\wedge_F^j(V \otimes_{E, \sigma_i} F)$ twisted by $D^{-1/2}$. We define an E_0 -linear representation D^c , $c \in \mathbb{Q}$ of $\mathfrak{u}(\Phi)$ in the E_0 -vector space E in exactly the same way as for $\mathfrak{gl}(m, F)$ above, i.e. by taking the trace of a matrix and multiplying it by $\frac{c}{m}$.

Consider the finite set Ω of F -vector spaces of the form $\wedge_1^{j_1} \otimes_F \dots \otimes_F \wedge_r^{j_r}$ for various $j_i \in \{1, \dots, m\}$. They are exactly the isomorphism classes of irreducible subrepresentations of the exterior tensor product of (twisted by $D^{-1/2}$ and extended to F) exterior algebra representations $\wedge_F^*(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F)$ of $\oplus_{i=1}^r (\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F) \cong \mathfrak{gl}(m, F)^{\oplus r}$.

Let $\wedge_F^*(V \otimes_{L, \sigma_i} F) \otimes_F (E \otimes_{L, \sigma_i} F) = \oplus_p S_p^i$ be the decomposition into irreducible components of the representation of $\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)$ obtained from the decompositions $E \otimes_{L, \sigma_i} F = (E \otimes_{E, \sigma_i} F) \oplus (E \otimes_{E, \bar{\sigma}_i} F) \cong D^{-1/2} \oplus D^{1/2} \cong F \oplus F$ and $V \otimes_{L, \sigma_i} F = (V \otimes_{E, \sigma_i} F) \oplus (V \otimes_{E, \bar{\sigma}_i} F)$ above.

Let Ω' be the finite set of F -vector subspaces of $(\wedge_F^*(V \otimes_{L, \sigma_1} F) \otimes_F (E \otimes_{L, \sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L, \sigma_r} F) \otimes_F (E \otimes_{L, \sigma_r} F))$ of the form $S_{p_1}^1 \otimes_F \dots \otimes_F S_{p_r}^r$ for various p_1, \dots, p_r . These subspaces are irreducible subrepresentations of the exterior tensor product of exterior algebra representations as a representation of $\oplus_{i=1}^r (\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F)$.

Galois group $S = \text{Gal}(F/k)$ acts on Ω by permuting factors in tensor products. It also acts on Ω' . Let $\Omega_1, \dots, \Omega_u$ be the orbits of S on Ω . For any i choose $(j_1, \dots, j_r) \in \Omega_i$ and define $U_i \subset (\wedge_F^*(V \otimes_{L, \sigma_1} F) \otimes_F (E \otimes_{L, \sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L, \sigma_r} F) \otimes_F (E \otimes_{L, \sigma_r} F))$ to be the sum of the elements of Ω' (as subspaces of $(\wedge_F^*(V \otimes_{L, \sigma_1} F) \otimes_F (E \otimes_{L, \sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L, \sigma_r} F) \otimes_F (E \otimes_{L, \sigma_r} F))$) lying in the S -orbit of any $S_{p_1}^1 \otimes_F \dots \otimes_F S_{p_r}^r$, which is isomorphic to $\wedge_1^{j_1} \otimes_F \dots \otimes_F \wedge_r^{j_r}$ as a $\oplus_{i=1}^r (\mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F) \cong \mathfrak{gl}(m, F)^{\oplus r}$ -module.

Then $U_i \subset (\wedge_F^*(V \otimes_{L, \sigma_1} F) \otimes_F (E \otimes_{L, \sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L, \sigma_r} F) \otimes_F (E \otimes_{L, \sigma_r} F))$ is an S -submodule and

$$(U_i)^S \cong \left(\bigoplus_{(j_1, \dots, j_r) \in \Omega_i} (\wedge_1^{j_1} \otimes_F \dots \otimes_F \wedge_r^{j_r})^{\oplus n_{j_1, \dots, j_r}} \right)^S$$

is a primary representation of \mathfrak{g} over k of dimension $\sum_{(j_1, \dots, j_r) \in \Omega_i} n_{j_1, \dots, j_r} \cdot \binom{m}{j_1} \cdot \dots \cdot \binom{m}{j_r}$. These representations $(U_i)^S$, $1 \leq i \leq u$ contain all representations of $\mathfrak{g} \otimes_k F$ of the form $\wedge_1^{j_1} \otimes_F \dots \otimes_F \wedge_r^{j_r}$ after extending scalars to F .

The reason why nontrivial multiplicities may appear is exactly the doubling $V \otimes_{L, \sigma_i} F = (V \otimes_{E, \sigma_i} F) \oplus (V \otimes_{E, \bar{\sigma}_i} F)$ described above. Hence one can compute multiplicities n_{j_1, \dots, j_r} as follows. Consider the finite set Ω'' of r -tuples of signs $+$ and $-$, i.e. $\Omega'' = \{(\alpha_1, \dots, \alpha_r) \mid \alpha_i = \pm 1\}$. Note that the i -th sign corresponds to the i -th embedding $\sigma_i: L \rightarrow F$ over k . Consider the action of $S = \text{Gal}(F/k)$ on Ω'' such that $g \in S$ acts on entries of r -tuples by the same permutations as on the set of left cosets S/\tilde{H} (where $\tilde{H} = \{g \in S \mid g \circ \sigma_1 = \sigma_1\}$) and g changes the sign in the i -th entry to the opposite sign (in the j -th entry, where $\sigma_j = g \circ \sigma_i$) if and only if $g(\theta) = -\theta$. Then

$$n_{j_1, \dots, j_r} = \frac{\text{order of the stabilizer of } (j_1, \dots, j_r) \in \Omega}{\text{order of the intersection of stabilizers of } (+, \dots, +) \in \Omega'' \text{ and of } (j_1, \dots, j_r) \in \Omega}.$$

This gives a description of some multiples of $(k = \mathbb{Q})$ -linear irreducible representations W_i of \mathfrak{g} mentioned in the Theorem above (as well as formulas for their dimensions - some multiples of $\dim_k(W_i)$) in terms of the Galois action.

6 Cohomology classes of division algebras.

In this section we compute division algebras D_i as elements of the Brauer group $Br(F/C_j) \cong H^2(\text{Gal}(F/C_j), F^*)$ as well as their centers C_j .

6.1 Case of the totally real field and odd dimension.

Let $E = E_0 = L$ be totally real and $m = \dim_E V$ odd. We saw above how to construct a primary representation $W = U^S$ of \mathfrak{g} over $k = \mathbb{Q}$, which contains irreducible representation $\rho^0 \boxtimes \dots \boxtimes \rho^0$ (the exterior tensor product of irreducible spin representations) of $\mathfrak{g} \otimes_k F \cong \bigoplus_{i=1}^r \mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$ after extending scalars to F . This means that $W \cong W_0^{\oplus \mu}$, where W_0 is an irreducible representation of \mathfrak{g} over k and $W_0 \otimes_k F \cong \frac{\dim_k W}{\mu \cdot (\dim_F(\rho^0))^r} \cdot \rho^0 \boxtimes \dots \boxtimes \rho^0$. Since we are interested only in the endomorphism algebra $D_0 = \text{End}_{\mathfrak{g}}(W_0)$ which is a central division algebra over k split over F , we can describe it by computing the Galois cohomology invariant of the central simple algebra $A = \text{End}_{\mathfrak{g}}(W) \cong \text{Mat}_{\mu \times \mu}(D_0)$, i.e. its Brauer invariant in $Br(F/k) \cong H^2(S, F^*)$, where $S = \text{Gal}(F/k)$. Then $\mu = \frac{\deg(A)}{\deg(D_0)} = \frac{n_0}{\deg(D_0)}$.

We will use the same notation as above with the following exceptions:

$$f_{\alpha_1, \dots, \alpha_l, \gamma} = (1 + \gamma \cdot f_0) \cdot f_{\alpha_1, 1} \cdot \dots \cdot f_{\alpha_l, l},$$

$$f_{\alpha \cdot i} = \left(e_i + \alpha \cdot \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right).$$

Some parts of our construction (in particular, the construction of the generators of endomorphism algebras) may be viewed as a generalization of some constructions of van Geemen [13], §3.

Consider F -linear homomorphisms

$$r_{((\alpha_i), \gamma), ((\beta_i), \tilde{\gamma})} : \tilde{C}(V \otimes_L F) \rightarrow I_{\beta_1, \dots, \beta_l, \tilde{\gamma}}, \quad \xi \mapsto \tau^{\delta(\gamma, \tilde{\gamma})}(\xi \cdot R_{((\alpha_i), \gamma), ((\beta_i), \tilde{\gamma})}),$$

where $\tau : C(V \otimes_L F) \rightarrow C(V \otimes_L F)$ is the algebra homomorphism induced by multiplication by (-1) on V , $\delta(\gamma, \tilde{\gamma}) = 1$, if $\gamma \neq \tilde{\gamma} \cdot (-1)^{P(\alpha, \beta)}$ (where $P(\alpha, \beta) = \text{card}\{i \mid \alpha_i \neq \beta_i\}$) and 0 otherwise, and

$$R_{((\alpha_i), \gamma), ((\beta_i), \tilde{\gamma})} = \frac{(-1)^{c(\alpha, \beta)}}{\prod_{i: \alpha_i = \beta_i} \Phi(f_i, f_{-i})} \cdot \prod_{i: \alpha_i = \beta_i} (f_{-\alpha_i \cdot i} \cdot f_{\alpha_i \cdot i}) \cdot \prod_{i: \alpha_i \neq \beta_i} f_{\beta_i \cdot i},$$

where $c(\alpha, \beta)$ is the number of transpositions of factors needed to transform the product $\prod_i f_{\alpha_i \cdot i} \cdot \prod_{i: \alpha_i \neq \beta_i} f_{\beta_i \cdot i}$ into the product $q \cdot \prod_i f_{\beta_i \cdot i}$ with some coefficient $q \in C(V \otimes_L F)$. Then $r_{((\alpha_i), \gamma), ((\beta_i), \tilde{\gamma})}$ is nonzero only on the factor $I_{\alpha_1, \dots, \alpha_l, \gamma}$ of $\tilde{C}(V \otimes_L F)$ and induces an isomorphism $I_{\alpha_1, \dots, \alpha_l, \gamma} \rightarrow I_{\beta_1, \dots, \beta_l, \tilde{\gamma}}$ which commutes with the action of $\mathfrak{so}(\Phi) \otimes_L F$.

In order to simplify notation we will denote index $((\alpha_i), \gamma)$ by α .

One can choose coefficients $\lambda_{\alpha, \beta} \in F^*$ such that under an isomorphism of F -algebras $\text{Mat}(F) \cong \text{End}_{\mathfrak{so}(\Phi) \otimes_L F}(\tilde{C}(V \otimes_L F))$ matrices of the form E_{ij} (in the notation of [1], §13) correspond to endomorphisms $\lambda_{\alpha, \beta} \cdot r_{\alpha, \beta}$. In order to do this, one can choose and fix index $\alpha^0 = ((\alpha_i^0), \gamma^0)$ and take

$$\lambda_{\alpha^0, \beta} = 1, \quad \lambda_{\beta, \alpha^0} = (-1)^{P(\alpha^0, \beta) \cdot \delta(\gamma^0, \tilde{\gamma}) + P(\alpha^0, \beta) \cdot (P(\alpha^0, \beta) - 1)/2} \cdot \prod_{i: \alpha_i^0 \neq \beta_i} \frac{1}{\Phi(f_i, f_{-i})}$$

and

$$\lambda_{\alpha, \beta} = \lambda_{\alpha, \alpha^0} \cdot (-1)^{e(\alpha, \beta) + \delta(\gamma, \tilde{\gamma}) \cdot (l + P(\alpha, \beta)) + \delta(\gamma, \gamma^0) \cdot (l + P(\alpha, \alpha^0)) + \delta(\gamma^0, \tilde{\gamma}) \cdot (l + P(\alpha^0, \beta))} \cdot \prod_{i: \alpha_i = \beta_i \neq \alpha_i^0} \Phi(f_i, f_{-i}),$$

where $\alpha = ((\alpha_i), \gamma)$, $\beta = ((\beta_i), \tilde{\gamma})$, $e(\alpha, \beta)$ is the number of transpositions of factors needed in order to transform the product $\prod_{i: \alpha_i^0 \neq \alpha_i} f_{\alpha_i \cdot i} \cdot \prod_{i: \alpha_i^0 \neq \beta_i} f_{-\beta_i \cdot i}$ into the product $\prod_{i: \alpha_i \neq \beta_i} f_{\alpha_i \cdot i} \cdot \prod_{i: \alpha_i = \beta_i \neq \alpha_i^0} (f_{\beta_i \cdot i} \cdot f_{-\beta_i \cdot i})$. Note that in this construction $\lambda_{\alpha, \beta} \in L^*$.

Then we construct endomorphisms

$$\begin{aligned} r_{(\alpha^i),(\beta^i)} &= r_{\alpha^1,\beta^1}^1 \circ \dots \circ r_{\alpha^r,\beta^r}^r : \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \rightarrow \\ &\rightarrow I_{\beta^1}^1 \otimes_F \dots \otimes_F I_{\beta^r}^r \subset \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \end{aligned}$$

which commute with $\mathfrak{g} \otimes_k F$, where $\alpha^p = ((\alpha_1^p, \dots, \alpha_l^p), \gamma^p)$, $\beta^p = ((\beta_1^p, \dots, \beta_l^p), \tilde{\gamma}^p)$ and

$$\begin{aligned} r_{\alpha^p,\beta^p}^p &= 1 \otimes_F \dots \otimes_F (r_{\alpha^p,\beta^p}) \otimes_F \dots \otimes_F 1 : \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \rightarrow \\ &\rightarrow \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \end{aligned}$$

(with 1 outside of the p -th spot).

As in [13], Proposition 3.6 F -algebra $End_{\mathfrak{g} \otimes_k F}(W \otimes_k F) = A \otimes_k F$ is generated by elements $r_{(\alpha^i),(\beta^i)}$ (more precisely, by those of them which correspond to the summands of $\tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F)$ included in $W \otimes_k F = U \subset \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F)$) or by elements $r_{\alpha,\beta}^p$, while k -algebra $A = End_{\mathfrak{g}}(W) = (A \otimes_k F)^S$ is generated by elements $r_{\alpha,\beta}^{p,q} = \sum_{g \in S} g(e_q) \cdot g \circ r_{\alpha,\beta}^p$, where $\{e_q\}$ is a basis of F/k .

Let us denote by $(c_{q,g})$ the inverse matrix of the matrix $(g(e_q))$. Then $r_{\alpha,\beta}^p = \sum_q c_{q,Id} \cdot r_{\alpha,\beta}^{p,q}$ and for any $g \in S = Gal(F/k)$ if we denote by $\phi_g : A \otimes_k F \rightarrow A \otimes_k F$ the conjugation by $g : a \otimes f \mapsto a \otimes g(f)$, then

$$\phi_g(r_{(\alpha^i),(\beta^i)}) = g \circ r_{(\alpha^i),(\beta^i)} = r_{g(\alpha^i),g(\beta^i)},$$

where the action of S on upper indices i (which number embeddings $\sigma_i : L \hookrightarrow F$) coincides with its action on left cosets S/\tilde{H} , where $\tilde{H} = \{g \in S \mid g|_{\sigma_1(L)} = Id_{\sigma_1(L)}\}$ and the action of $g \in S$ on indices $\alpha = ((\alpha_1, \dots, \alpha_l), \gamma)$ is given by the rule $g(\alpha) = ((c_1(g) \cdot \alpha_1, \dots, c_l(g) \cdot \alpha_l), c_0(g) \cdot \gamma)$, where $c_i(g) \in \{\pm 1\}$ and $g(f_{\alpha_1, \dots, \alpha_l, \gamma}) = f_{c_1(g) \cdot \alpha_1, \dots, c_l(g) \cdot \alpha_l, c_0(g) \cdot \gamma}$.

Hence the matrix of $m(g) \in GL(W \otimes_k F)$ is such that

$$m(g) \cdot E_{i,j} \cdot m(g)^{-1} = \phi_g(E_{i,j}) = \left(\prod_{i=1}^r \frac{g(\lambda_{\alpha^i, \beta^i})}{\lambda_{g(\alpha^i), g(\beta^i)}} \right) \cdot E_{g(i), g(j)},$$

where $E_{i,j}$ denotes a matrix from $Mat(F) \cong End_{\mathfrak{g} \otimes_k F}(W \otimes_k F)$ corresponding to $r_{(\alpha^i),(\beta^i)}$, i.e. upto a scalar multiple conjugation by $m(g)$ acts on matrices as the (same) permutation of columns and rows induced by g on indices $((\alpha_j^i), \gamma^i)$.

Then the element of $H^2(S, F^*)$ corresponding to the central division algebra $D_0 = End_{\mathfrak{g}}(W_0)$ is the class of a 2-cocycle $\lambda : S \times S \rightarrow F^* \cong F^* \cdot Id \subset Mat(F)$, $(g_1, g_2) \mapsto m(g_1 g_2) \cdot (g_1(m(g_2)))^{-1} \cdot m(g_1)^{-1}$ [8], [5].

6.2 Case of the totally real field and even dimension.

Let $E = E_0 = L$ be totally real and $m = \dim_E V$ even. We saw above how to construct a primary representation $W = (U_i)^S$ of \mathfrak{g} over $k = \mathbb{Q}$, which contains irreducible representation $\rho^{\alpha_1} \boxtimes \dots \boxtimes \rho^{\alpha_r}$ (the exterior tensor product of irreducible semi-spin representations) of $\mathfrak{g} \otimes_k F \cong \oplus_{i=1}^r \mathfrak{so}(\Phi) \otimes_{L, \sigma_i} F$ after extending scalars to F (as well as its Galois conjugates). This means that $W \cong W_0^{\oplus \mu}$, where W_0 is an irreducible representation of \mathfrak{g} over k , $W \otimes_k F \cong \oplus_i W_i$ and $W_i \cong \frac{\dim_F W_i}{(\dim_F(\rho^{\alpha_1}))^r} \cdot \rho^{\alpha_1'} \boxtimes \dots \boxtimes \rho^{\alpha_r'}$ are the isotypical components (over F). Since we are interested only in the endomorphism algebra $D_0 = \text{End}_{\mathfrak{g}}(W_0)$ which is a division algebra over k (and over its center C) split over F , we can describe it by computing the Galois cohomology invariant of the central simple algebra $A = \text{End}_{\mathfrak{g}}(W) \cong \text{Mat}_{\mu \times \mu}(D_0)$ (over C), i.e. its Brauer invariant in $Br(F/C) \cong H^2(S', F^*)$, where $S' = \text{Gal}(F/C)$. Then $\mu = \frac{\deg(A)}{\deg(D_0)} = \frac{n_{\alpha_1, \dots, \alpha_r}}{\deg(D_0)}$.

We will use the same notation as above with the following exceptions:

$$f_{\alpha_1, \dots, \alpha_l} = f_{\alpha_1 \cdot 1} \cdot \dots \cdot f_{\alpha_l \cdot l},$$

$$f_{\alpha \cdot i} = \left(e_i + \alpha \cdot \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right).$$

Some parts of our construction (in particular, the construction of the generators of endomorphism algebras) may be viewed as a generalization of some constructions of van Geemen [13], §3.

Consider F -linear homomorphisms

$$r_{(\alpha_i), (\beta_i)} : \tilde{C}(V \otimes_L F) \rightarrow I_{\beta_1, \dots, \beta_l}, \quad \xi \mapsto \xi \cdot R_{(\alpha_i), (\beta_i)},$$

where $P(\alpha, \beta) = \text{card}\{i \mid \alpha_i \neq \beta_i\}$ and

$$R_{(\alpha_i), (\beta_i)} = \frac{(-1)^{c(\alpha, \beta)}}{\prod_{i: \alpha_i = \beta_i} \Phi(f_i, f_{-i})} \cdot \prod_{i: \alpha_i = \beta_i} (f_{-\alpha_i \cdot i} \cdot f_{\alpha_i \cdot i}) \cdot \prod_{i: \alpha_i \neq \beta_i} f_{\beta_i \cdot i},$$

where $c(\alpha, \beta)$ is the number of transpositions of factors needed to transform the product $\prod_i f_{\alpha_i \cdot i} \cdot \prod_{i: \alpha_i \neq \beta_i} f_{\beta_i \cdot i}$ into the product $q \cdot \prod_i f_{\beta_i \cdot i}$ with some coefficient $q \in C(V \otimes_L F)$. Then $r_{(\alpha_i), (\beta_i)}$ is nonzero only on the factor $I_{\alpha_1, \dots, \alpha_l}$ of $\tilde{C}(V \otimes_L F)$ and induces an isomorphism $I_{\alpha_1, \dots, \alpha_l} \rightarrow I_{\beta_1, \dots, \beta_l}$ which commutes with action of $\mathfrak{so}(\Phi) \otimes_L F$. Without mentioning this explicitly, we will be restricting all our endomorphisms to the factors of $\tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$ contributing to an isotypical component $W_i \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$.

In order to simplify notation we will denote index (α_i) by α .

One can choose coefficients $\lambda_{\alpha, \beta} \in F^*$ such that under an isomorphism of F -algebras $\text{Mat}(F) \cong \text{End}_{\mathfrak{so}(\Phi) \otimes_L F}(W_i)$ (note that $W_i \subset \tilde{C}(V \otimes_{L, \sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L, \sigma_r} F)$) and see

the remark above) matrices of the form E_{ij} correspond to endomorphisms $\lambda_{\alpha,\beta} \cdot r_{\alpha,\beta}$. In order to do this, one can choose and fix index $\alpha^0 = (\alpha_i^0)$ and take

$$\lambda_{\alpha^0,\beta} = 1, \quad \lambda_{\beta,\alpha^0} = (-1)^{P(\alpha^0,\beta) \cdot (P(\alpha^0,\beta)-1)/2} \cdot \prod_{i: \alpha_i^0 \neq \beta_i} \frac{1}{\Phi(f_i, f_{-i})}$$

and

$$\lambda_{\alpha,\beta} = \lambda_{\alpha,\alpha^0} \cdot (-1)^{e(\alpha,\beta)} \cdot \prod_{i: \alpha_i = \beta_i \neq \alpha_i^0} \Phi(f_i, f_{-i}),$$

where $\alpha = (\alpha_i)$, $\beta = (\beta_i)$, $e(\alpha, \beta)$ is the number of transpositions of factors needed in order to transform the product $\prod_{i: \alpha_i^0 \neq \alpha_i} f_{\alpha_i \cdot i} \cdot \prod_{i: \alpha_i^0 \neq \beta_i} f_{-\beta_i \cdot i}$ into the product $\prod_{i: \alpha_i \neq \beta_i} f_{\alpha_i \cdot i} \cdot \prod_{i: \alpha_i = \beta_i \neq \alpha_i^0} (f_{\beta_i \cdot i} \cdot f_{-\beta_i \cdot i})$. Note that in this construction $\lambda_{\alpha,\beta} \in L^*$.

Then we construct endomorphisms

$$\begin{aligned} r_{(\alpha^i),(\beta^i)} &= r_{\alpha^1,\beta^1}^1 \circ \dots \circ r_{\alpha^r,\beta^r}^r : \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \rightarrow \\ &\rightarrow I_{\beta^1}^1 \otimes_F \dots \otimes_F I_{\beta^r}^r \subset \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \end{aligned}$$

which commute with $\mathfrak{g} \otimes_k F$, where $\alpha^p = (\alpha_1^p, \dots, \alpha_l^p)$, $\beta^p = (\beta_1^p, \dots, \beta_l^p)$ and

$$\begin{aligned} r_{\alpha^p,\beta^p}^p &= 1 \otimes_F \dots \otimes_F (r_{\alpha^p,\beta^p}) \otimes_F \dots \otimes_F 1 : \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \rightarrow \\ &\rightarrow \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F) \end{aligned}$$

(with 1 outside of the p -th spot).

As in [13], Proposition 3.6 F -algebra $End_{\mathfrak{g} \otimes_k F}(W \otimes_k F) = A \otimes_k F$ is generated by elements $r_{(\alpha^i),(\beta^i)}$ (more precisely, by those of them which correspond to the summands of $\tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F)$ included in various isotypical components $W_{i'} \otimes_k F \subset U_i \subset \tilde{C}(V \otimes_{L,\sigma_1} F) \otimes_F \dots \otimes_F \tilde{C}(V \otimes_{L,\sigma_r} F)$) or by elements $r_{\alpha,\beta}^p$, while k -algebra $A = End_{\mathfrak{g}}(W) = (A \otimes_k F)^S$ is generated by elements $r_{\alpha,\beta}^{p,q} = \sum_{g \in S} g(e_q) \cdot g \circ r_{\alpha,\beta}^p$, where $\{e_q\}$ is a basis of F/k .

The center C of A (and of D_0) consists of Galois averages (as above) of F -linear combinations of sums $C_{i'} = \sum_{(\alpha^j) \in I_{i'}} (\prod_{i=1}^r \sigma_i(\lambda_{\alpha^i, \alpha^i})) \cdot r_{(\alpha^j),(\alpha^j)}$ (over the sets $I_{i'}$ of indices α^j corresponding to irreducible subrepresentations over F of $W \otimes_k F$ contained in various isotypical components $W_{i'}$). Each of the coefficients of these F -linear combinations gives a field embedding $C \rightarrow F$ over $k = \mathbb{Q}$. Note that $A \otimes_k F \cong \prod A \otimes_C F$, where the product is taken over these embeddings (which are numbered by the isotypical components $W_{i'}$ of $W \otimes_k F$ over F) and $A \otimes_C F \cong End_{\mathfrak{g} \otimes_k F}(W_{i'})$. Moreover, the projection $A \otimes_k F \rightarrow A \otimes_C F$ is given by annihilating endomorphisms between irreducible subrepresentations of isotypical components $W_{i''}$ different from $W_{i'}$. More explicitly the subfield $C \subset F$ under the embedding corresponding to an isotypical component $W_{i'}$ is the fixed subfield of the subgroup $S' \subset S$ consisting of those $g \in S$ which preserve the isotypical component: $g(W_{i'}) = W_{i'}$. Let us choose one such embedding $C \rightarrow F$ (which corresponds to a choice of an isotypical

component $W_{i'}$ of $W \otimes_k F$).

Let us denote by $(c_{q,g})$ the inverse matrix of the matrix $(g(e_q))$. Then $r_{\alpha,\beta}^p = \sum_q c_{q,Id} \cdot r_{\alpha,\beta}^{p,q}$ and for any $g \in S' = \text{Gal}(F/C) \subset S = \text{Gal}(F/k)$ if we denote by $\phi_g: A \otimes_C F \rightarrow A \otimes_C F$ the conjugation by $g: a \otimes f \mapsto a \otimes g(f)$, then

$$\phi_g(r_{(\alpha^i),(\beta^i)}) = g \circ r_{(\alpha^i),(\beta^i)} = r_{g(\alpha^i),g(\beta^i)},$$

where the action of $S' \subset S$ on upper indices i (which number embeddings $\sigma_i: L \hookrightarrow F$) coincides with its action on left cosets S/\tilde{H} , where $\tilde{H} = \{g \in S \mid g|_{\sigma_1(L)} = \text{Id}_{\sigma_1(L)}\}$ and the action of $g \in S' \subset S$ on indices $\alpha = (\alpha_1, \dots, \alpha_l)$ is given by the rule $g(\alpha) = (c_1(g) \cdot \alpha_1, \dots, c_l(g) \cdot \alpha_l)$, where $c_i(g) \in \{\pm 1\}$ and $g(f_{\alpha_1, \dots, \alpha_l}) = f_{c_1(g) \cdot \alpha_1, \dots, c_l(g) \cdot \alpha_l}$.

Hence the matrix of $m(g) \in GL(W_{i'})$ is such that

$$m(g) \cdot E_{i,j} \cdot m(g)^{-1} = \phi_g(E_{i,j}) = \left(\prod_{i=1}^r \frac{g(\lambda_{\alpha^i, \beta^i})}{\lambda_{g(\alpha^i), g(\beta^i)}} \right) \cdot E_{g(i), g(j)},$$

where $E_{i,j}$ denotes a matrix from $\text{Mat}(F) \cong \text{End}_{\mathfrak{g} \otimes_k F}(W_{i'})$ corresponding to $r_{(\alpha^i),(\beta^i)}$, i.e. upto a scalar multiple conjugation by $m(g)$ acts on matrices as the (same) permutation of columns and rows induced by g on indices (α_j^i) .

Then the element of $H^2(S', F^*)$ corresponding to the central division algebra $D_0 = \text{End}_{\mathfrak{g}}(W_0)$ (over C) is the class of a 2-cocycle $\lambda: S' \times S' \rightarrow F^* \cong F^* \cdot \text{Id} \subset \text{Mat}(F)$, $(g_1, g_2) \mapsto m(g_1 g_2) \cdot (g_1(m(g_2)))^{-1} \cdot m(g_1)^{-1}$ [8], [5].

6.3 Case of the CM-field.

Let $E = E_0(\theta)$, $\theta^2 \in E_0 = L$ be a CM-field and $m = \dim_E V$. We saw above how to construct a primary representation $W = (U_i)^S$ of \mathfrak{g} over $k = \mathbb{Q}$, which contains the irreducible representation $\rho_{j_1}^{\alpha_1} \boxtimes \dots \boxtimes \rho_{j_r}^{\alpha_r}$ of $\mathfrak{g} \otimes_k F \cong \oplus_{i=1}^r \mathfrak{u}(\Phi) \otimes_{L, \sigma_i} F \cong \mathfrak{gl}(m, F)^{\oplus r}$ after extending scalars to F (as well as its Galois conjugates). Here $\alpha_i \in \{\pm 1\}$, $1 \leq j_i \leq m$ and

$$\rho_{j_i}^{\alpha_i}: \mathfrak{gl}(m, F) \rightarrow \text{End}_F(\wedge_F^{j_i}(V \otimes_{E, \alpha_i \cdot \sigma} F) \otimes_F F)$$

is the exterior product representation twisted by $D^{\alpha_i/2}$, where $\pm \sigma: E \rightarrow F$ are the two embeddings extending $\sigma: L \rightarrow F$. This means that $W \cong W_0^{\oplus \mu}$, where W_0 is an irreducible representation of \mathfrak{g} over k , $W \otimes_k F \cong \oplus_i W_i$ and $W_i \cong \frac{\dim_F W_i}{\dim_F(\rho_{j_1}^{\alpha_1} \dots \rho_{j_r}^{\alpha_r})} \cdot \rho_{j_1}^{\alpha_1'} \boxtimes \dots \boxtimes \rho_{j_r}^{\alpha_r'}$ are the isotypical components (over F). Since we are interested only in the endomorphism algebra $D_0 = \text{End}_{\mathfrak{g}}(W_0)$ which is a division algebra over k (and over its center C) split over F , we can describe it by computing the Galois cohomology invariant of the central simple algebra $A = \text{End}_{\mathfrak{g}}(W) \cong \text{Mat}_{\mu \times \mu}(D_0)$ (over C), i.e. its Brauer invariant in $Br(F/C) \cong H^2(S', F^*)$, where $S' = \text{Gal}(F/C)$. Then $\mu = \frac{\deg(A)}{\deg(D_0)} = \frac{n_{j_1, \dots, j_r}}{\deg(D_0)}$.

Our computation is analogous to the case of a totally real field considered above.

Consider F -linear homomorphisms

$$r_{\alpha,\beta}: \wedge_F^*(V \otimes_{E,\alpha\cdot\sigma} F) \otimes_F F \rightarrow \wedge_F^*(V \otimes_{E,\beta\cdot\sigma} F) \otimes_F F, \xi \mapsto (\tau_*)^{P(\alpha,\beta)}(\xi),$$

where $P(-1, +1) = 1$, $P(+1, -1) = -1$, $P(\alpha, \alpha) = 0$ and $\tau_* = \oplus_p \tau_p$ is the direct sum of isomorphisms of $\mathfrak{gl}(m, F)$ -modules

$$\wedge_F^p(V \otimes_{E,\sigma_i} F) \otimes_F (E \otimes_{E,\sigma_i} F) \rightarrow \wedge_F^{m-p}(V \otimes_{E,\sigma_i} F) \otimes_F D^{-1/2}$$

introduced above. Then $r_{\alpha,\beta}$ induces an isomorphism

$$\wedge_F^{j_i}(V \otimes_{E,\alpha\cdot\sigma} F) \otimes_F F \rightarrow \wedge_F^{j'_i}(V \otimes_{E,\beta\cdot\sigma} F) \otimes_F F$$

which commutes with the action of $\mathfrak{u}(\Phi) \otimes_L F \cong \mathfrak{gl}(m, F)$. Without mentioning this explicitly, we will be restricting all our endomorphisms to the factors of $(\wedge_F^*(V \otimes_{L,\sigma_1} F) \otimes_F (E \otimes_{L,\sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L,\sigma_r} F) \otimes_F (E \otimes_{L,\sigma_r} F))$ contributing to an isotypical component $W_i \subset (\wedge_F^*(V \otimes_{L,\sigma_1} F) \otimes_F (E \otimes_{L,\sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L,\sigma_r} F) \otimes_F (E \otimes_{L,\sigma_r} F))$.

Then we construct endomorphisms

$$\begin{aligned} r_{(\alpha^i),(\beta^i)} &= r_{\alpha^1,\beta^1}^1 \circ \dots \circ r_{\alpha^r,\beta^r}^r: (\wedge_F^*(V \otimes_{L,\sigma_1} F) \otimes_F (E \otimes_{L,\sigma_1} F)) \otimes_F \dots \\ &\quad \dots \otimes_F (\wedge_F^*(V \otimes_{L,\sigma_r} F) \otimes_F (E \otimes_{L,\sigma_r} F)) \rightarrow \\ &\rightarrow (\wedge_F^*(V \otimes_{L,\sigma_1} F) \otimes_F (E \otimes_{L,\sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L,\sigma_r} F) \otimes_F (E \otimes_{L,\sigma_r} F)), \end{aligned}$$

which commute with $\mathfrak{g} \otimes_k F$, where $(\alpha^i) = (\alpha^1, \dots, \alpha^r)$, $(\beta^i) = (\beta^1, \dots, \beta^r)$ and

$$\begin{aligned} r_{\alpha^p,\beta^p}^p &= 1 \otimes_F \dots \otimes_F (r_{\alpha^p,\beta^p}) \otimes_F \dots \otimes_F 1: (\wedge_F^*(V \otimes_{L,\sigma_1} F) \otimes_F (E \otimes_{L,\sigma_1} F)) \otimes_F \dots \\ &\quad \dots \otimes_F (\wedge_F^*(V \otimes_{L,\sigma_r} F) \otimes_F (E \otimes_{L,\sigma_r} F)) \rightarrow \\ &\rightarrow (\wedge_F^*(V \otimes_{L,\sigma_1} F) \otimes_F (E \otimes_{L,\sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L,\sigma_r} F) \otimes_F (E \otimes_{L,\sigma_r} F)) \end{aligned}$$

(with 1 outside of the p -th spot).

As in the case of a totally real field E , F -algebra $End_{\mathfrak{g} \otimes_k F}(W \otimes_k F) = A \otimes_k F$ is generated by elements $r_{(\alpha^i),(\beta^i)}$ (more precisely, by those of them which correspond to the summands of $(\wedge_F^*(V \otimes_{L,\sigma_1} F) \otimes_F (E \otimes_{L,\sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L,\sigma_r} F) \otimes_F (E \otimes_{L,\sigma_r} F))$ included in various isotypical components $W_{i'} \otimes_k F \subset U_i \subset (\wedge_F^*(V \otimes_{L,\sigma_1} F) \otimes_F (E \otimes_{L,\sigma_1} F)) \otimes_F \dots \otimes_F (\wedge_F^*(V \otimes_{L,\sigma_r} F) \otimes_F (E \otimes_{L,\sigma_r} F))$) or by elements $r_{\alpha,\beta}^p$, while k -algebra $A = End_{\mathfrak{g}}(W) = (A \otimes_k F)^S$ is generated by elements $r_{\alpha,\beta}^{p,q} = \sum_{g \in S} g(e_q) \cdot g \circ r_{\alpha,\beta}^p$, where $\{e_q\}$ is a basis of F/k .

The center C of A (and of D_0) can be computed exactly as in the case of a totally real field. In particular, field embeddings $C \rightarrow F$ correspond to the isotypical components $W_{i'}$ of $W \otimes_k F$ over F , $A \otimes_C F \cong End_{\mathfrak{g} \otimes_k F}(W_{i'})$, the projection $A \otimes_k F \cong \prod A \otimes_C F \rightarrow A \otimes_C F$ is given by annihilating endomorphisms between irreducible subrepresentations of isotypical

components $W_{i''}$ different from $W_{i'}$ and the subfield $C \subset F$ under the embedding corresponding to an isotypical component $W_{i'}$ is the fixed subfield of the subgroup $S' \subset S$ consisting of those $g \in S$ which preserve the isotypical component: $g(W_{i'}) = W_{i'}$. Let us choose one such embedding $C \rightarrow F$ (which corresponds to a choice of an isotypical component $W_{i'}$ of $W \otimes_k F$).

Let us denote by $(c_{q,g})$ the inverse matrix of the matrix $(g(e_q))$. Then $r_{\alpha,\beta}^p = \sum_q c_{q,Id} \cdot r_{\alpha,\beta}^{p,q}$ and for any $g \in S' = \text{Gal}(F/C) \subset S = \text{Gal}(F/k)$ if we denote by $\phi_g: A \otimes_C F \rightarrow A \otimes_C F$ the conjugation by $g: a \otimes f \mapsto a \otimes g(f)$, then

$$\phi_g(r_{(\alpha^i),(\beta^i)}) = g \circ r_{(\alpha^i),(\beta^i)} = \left(\prod_k \lambda_{\alpha^k,\beta^k}(g) \right) \cdot r_{g(\alpha^i),g(\beta^i)},$$

where the action of $S' \subset S$ on upper indices i (which number embeddings $\sigma_i: L \hookrightarrow F$) coincides with its action on the left cosets S/\tilde{H} , where $\tilde{H} = \{g \in S \mid g|_{\sigma_1(L)} = \text{Id}_{\sigma_1(L)}\}$ and moreover $g \in S' \subset S$ multiplies the i -th index α^i in the r -tuple $(\alpha^i) = (\alpha^1, \dots, \alpha^r)$ by $g(\theta)/\theta = \pm 1$.

Here $\lambda_{\alpha^k,\beta^k}(g) \in F^*$ are suitable constants. In order to compute them, note that isomorphisms

$$\begin{aligned} \tau_p: \wedge_F^p(V \otimes_{E,\sigma_i} F) \otimes_F (E \otimes_{E,\sigma_i} F) &\rightarrow \wedge_F^p(V \otimes_{E,\sigma_i} F) \otimes_F (E \otimes_{E,\sigma_i} F) \cong \\ &\cong \wedge_F^p((V \otimes_{E,\sigma_i} F)^*) \otimes_F (E \otimes_{E,\sigma_i} F)^* \rightarrow \wedge_F^{m-p}(V \otimes_{E,\sigma_i} F) \otimes_F (E \otimes_{E,\sigma_i} F) \end{aligned}$$

(where the first arrow is the isomorphism determined by the matrix of Φ^{-1}) are defined over E . If we assume that the isomorphism $\wedge_E^p(V)^* \rightarrow \wedge_E^{m-p}(V) \otimes_E E$ is defined via the pairing

$$\wedge_E^p(V) \otimes_E \wedge_E^{m-p}(V) \rightarrow \wedge_E^m(V) \cong E, \quad x \otimes y \mapsto x \wedge y,$$

then we find that $\lambda_{\alpha^k,\beta^k}(g) = 1$, if $g(\theta) = \theta$ or $\alpha^k = \beta^k$ and $\lambda_{\alpha^k,\beta^k}(g) = (-1)^{p(m-p)} \cdot (g(\sigma_k(\text{disc}(\Phi))))^{-P(\alpha^k,\beta^k)}$ otherwise.

Hence the matrix of $m(g) \in GL(W_{i'})$ is such that

$$m(g) \cdot E_{i,j} \cdot m(g)^{-1} = \phi_g(E_{i,j}) = \left(\prod_k \lambda_{\alpha^k,\beta^k}(g) \right) E_{g(i),g(j)},$$

where $E_{i,j}$ denotes a matrix from $\text{Mat}(F) \cong \text{End}_{\mathfrak{g} \otimes_k F}(W_{i'})$ corresponding to $r_{(\alpha^i),(\beta^i)}$, i.e. conjugation by $m(g)$ acts on matrices upto a constant as the (same) permutation of columns and rows induced by g on indices (α^i) .

Then the element of $H^2(S', F^*)$ corresponding to the central division algebra $D_0 = \text{End}_{\mathfrak{g}}(W_0)$ (over C) is the class of a 2-cocycle $\lambda: S' \times S' \rightarrow F^* \cong F^* \cdot \text{Id} \subset \text{Mat}(F)$, $(g_1, g_2) \mapsto m(g_1 g_2) \cdot (g_1(m(g_2)))^{-1} \cdot m(g_1)^{-1}$ [8], [5].

7 Example.

Let $k = \mathbb{Q}$, $r = 3$ and $5 \leq m \leq 6$. Let $\rho < 0$ be the negative root of the cubic polynomial $f(t) = t^3 - 3t + 1$. Then $\frac{1}{1-\rho}$ and $1 - \frac{1}{\rho}$ are the other two roots of $f(t)$ and $E = L = k(\rho)$ is a totally real cyclic cubic Galois number field [6].

Let $\Phi = -\rho \cdot X_1^2 - \rho \cdot X_2^2 - X_3^2 - \dots - X_m^2$. Then by [9] there is a $K3$ surface X such that $\text{End}_{Hdg}(V) \cong E$ (where V is the \mathbb{Q} -lattice of transcendental cycles on X), $\dim_E V = m$ and $\Phi: V \otimes_E V \rightarrow E$ is the quadratic form constructed in [15].

Let $F = k\left(\sqrt{\rho}, \sqrt{\frac{1}{1-\rho}}, \sqrt{1 - \frac{1}{\rho}}\right)$ be our choice of a splitting field. Note that $L \subset F$ and $\sqrt{-1} = \sqrt{\rho} \cdot \sqrt{\frac{1}{1-\rho}} \cdot \sqrt{1 - \frac{1}{\rho}} \in F$. Then

$$S = \text{Gal}(F/k) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \rtimes \mathbb{Z}/3\mathbb{Z}$$

is a nonabelian extension of $\mathbb{Z}/3\mathbb{Z} \cong \text{Gal}(L/k)$ with generator g by $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$ with generators h_1, h_2, h_3 , where g acts on the generators (h_1, h_2, h_3) by the permutation (123) . We also denote by g the element of S such that $g(\sqrt{\rho}) = \sqrt{\frac{1}{1-\rho}}$, $g\left(\sqrt{\frac{1}{1-\rho}}\right) = \sqrt{1 - \frac{1}{\rho}}$, $g\left(\sqrt{1 - \frac{1}{\rho}}\right) = \sqrt{\rho}$. We assume that each generator h_i , $1 \leq i \leq 3$ multiplies by -1 the i -th square root among $\sqrt{\rho}, \sqrt{\frac{1}{1-\rho}}, \sqrt{1 - \frac{1}{\rho}}$ and does not change the others and that $h_i|_L = Id$.

There are 3 field embeddings $L \hookrightarrow F$: $\sigma_1 = Id$, $\sigma_2 = g|_L$ and $\sigma_3 = g^2|_L$. Then $\sqrt{\sigma_1(d_1)} = \sqrt{\sigma_1(d_2)} = \sqrt{-1} \cdot \sqrt{\rho}$, $\sqrt{\sigma_2(d_1)} = \sqrt{\sigma_2(d_2)} = \sqrt{-1} \cdot \sqrt{\frac{1}{1-\rho}}$, $\sqrt{\sigma_3(d_1)} = \sqrt{\sigma_3(d_2)} = \sqrt{-1} \cdot \sqrt{1 - \frac{1}{\rho}}$, $\sqrt{\sigma_1(d_3)} = \sqrt{\sigma_2(d_3)} = \sqrt{\sigma_3(d_3)} = \sqrt{-1}$, $\sqrt{-\sigma_i(d_{m-j+1})} = 1$ for any $i = 1, 2, 3$, $1 \leq j \leq l = [\frac{m}{2}]$. Hence $\otimes_{L, \sigma_1} \Gamma_1 = \otimes_{L, \sigma_1} \Gamma_2 = \sqrt{-1} \cdot \sqrt{\rho}$, $\otimes_{L, \sigma_2} \Gamma_1 = \otimes_{L, \sigma_2} \Gamma_2 = \sqrt{-1} \cdot \sqrt{\frac{1}{1-\rho}}$, $\otimes_{L, \sigma_3} \Gamma_1 = \otimes_{L, \sigma_3} \Gamma_2 = \sqrt{-1} \cdot \sqrt{1 - \frac{1}{\rho}}$, $\otimes_{L, \sigma_i} \Gamma_3 = \sqrt{-1}$ for all i (if $m = 6$).

(1) Let us consider first the case $m = 5$. The root system is of type B_2 : $R_0 = \{\pm \epsilon_p, \pm \epsilon_p \pm \epsilon_q \mid p, q = 1, 2\}$ with basis $B_0 = \{\epsilon_1 - \epsilon_2, \epsilon_2\}$. Hence $B_i = \{\epsilon_1 \otimes_{L, \sigma_i} \Gamma_1 - \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2, \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2\}$, $1 \leq i \leq 3$. The restriction of the spin representation of $\mathfrak{so}(\phi) \otimes_k F$ in $C^+(V \otimes_k F)$ to $\mathfrak{g} \otimes_k F = \text{Res}_{L/k}(\mathfrak{so}(\Phi)) \otimes_k F$ is isomorphic over F to 2^8 copies of the exterior tensor product $\rho^0 \boxtimes \rho^0 \boxtimes \rho^0$ of the irreducible spin representation of $\mathfrak{so}(\Phi) \otimes_L F$. Hence over $k = \mathbb{Q}$ the restriction of the spin representation of $\mathfrak{so}(\phi)$ in $C^+(V)$ to $\mathfrak{g} = \text{Res}_{L/k}(\mathfrak{so}(\Phi)) \subset \mathfrak{so}(\phi)$ is one single irreducible representation with multiplicity μ which splits over F into $\frac{2^8}{\mu}$ copies of $\rho^0 \boxtimes \rho^0 \boxtimes \rho^0$: $C^+(V) \cong U^{\oplus \mu}$.

In order to estimate $\frac{2^8}{\mu}$ (which divides n_0), let us consider

$$f_{1, \dots, 1, 1} = f_1 \cdot \dots \cdot f_l \cdot (1 + f_0) = q \cdot \prod_{i=1}^l \left(e_i + \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right) \cdot \left(1 + \frac{1}{\sqrt{d_{l+1}}} \cdot e_{l+1} \right)$$

(we use notation as above), where $q \in F$ is such that $\sigma(q) = \pm q$ for any $\sigma \in S = \text{Gal}(F/k)$. In our case

$$f_{1,\dots,1,1} = q \cdot (e_1 + \sqrt{-1} \cdot \sqrt{\rho} \cdot e_5) \cdot (e_2 + \sqrt{-1} \cdot \sqrt{\rho} \cdot e_4) \cdot (1 - \sqrt{-1} \cdot e_3).$$

Hence the stabilizer of (the line in $C(V \otimes_L F)$ generated by) $f_{1,\dots,1,1}$ consists of the elements g^k , i.e. has order 3. Since $\text{Gal}(F/k)$ has 24 elements total, we find that $n_0 = 8$. Hence either $\frac{2^8}{\mu} = 1$ or $\frac{2^8}{\mu} = 2$ or $\frac{2^8}{\mu} = 4$ or $\frac{2^8}{\mu} = 8$. In the first case, $\rho^0 \boxtimes \rho^0 \boxtimes \rho^0$ is already defined over \mathbb{Q} and $\mu = 2^8$, while in the other cases $\mu = 2^7$, $\mu = 2^6$ and $\mu = 2^5$ respectively.

Hence in this case $\text{End}(KS(X))_{\mathbb{Q}} \cong \text{Mat}_{\mu \times \mu}(D)$, where $D = \text{End}_{\mathfrak{g}}(U)$ is a division algebra. Let us check that $D \cong \mathbb{Q}$.

Let us compute the cohomological invariant of D . In our case

$$W \otimes_k F = V_{(1,1,1)} \oplus V_{(1,1',1')} \oplus V_{(1',1,1')} \oplus V_{(1',1',1)} \oplus V_{(2',2,2)} \oplus V_{(2,2',2)} \oplus V_{(2,2,2')} \oplus V_{(2',2',2')},$$

where $V_{(p_1,p_2,p_3)} = S_{p_1}^1 \otimes_F S_{p_2}^2 \otimes_F S_{p_3}^3$ in the notation of Section 5.2 and the values $1, 1', 2, 2'$ of p_i correspond to the indices $(\alpha_1, \alpha_2, \gamma)$ of ideals $I_{\alpha_1, \alpha_2, \gamma}$ as follows: $1 = (+++)$, $1' = (- - +)$, $2 = (- - -)$, $2' = (+ + -)$.

Let us denote $\bar{1} = 2$, $\bar{1}' = 2'$, $\bar{2} = 1$, $\bar{2}' = 1'$ and $\tilde{1} = 2'$, $\tilde{1}' = 2$, $\tilde{2} = 1'$, $\tilde{2}' = 1$. Then $g(V_{(p_1,p_2,p_3)}) = V_{(p_3,p_1,p_2)}$ and $h_i(V_{(p_1,p_2,p_3)}) = V_{(q_1,q_2,q_3)}$, where $q_i = \tilde{p}_i$ and $q_j = \bar{p}_j$ for $j \neq i$.

Let us denote $a = (1, 1', 1')$, $b = (1', 1, 1')$, $c = (1', 1', 1)$, $d = (1, 1, 1)$, $p = (2', 2, 2)$, $q = (2, 2', 2)$, $r = (2, 2, 2')$, $s = (2', 2', 2')$. Then using formulas from Section 6 we can choose coefficients $\lambda_{\alpha,\beta} = \prod_{i=1}^r \lambda_{\alpha^i, \beta^i} \in F^*$ as follows:

- $\lambda_{\alpha,\beta} = 1$ for $(\alpha, \beta) \in \{(d, -), (s, -), (a, a), (b, b), (c, c), (p, p), (q, q), (r, r)\}$,
- $\lambda_{\alpha,\beta} = 1$ for $(\alpha, \beta) \in \{(b, q), (a, p), (c, r), (q, b), (p, a), (r, c)\}$,
- $\lambda_{\alpha,\beta} = c_1$ for $(\alpha, \beta) \in \{(b, a), (b, p), (c, a), (c, p), (q, a), (q, p), (r, a), (r, p)\}$,
- $\lambda_{\alpha,\beta} = c_2$ for $(\alpha, \beta) \in \{(a, b), (a, q), (c, b), (c, q), (p, b), (p, q), (r, b), (r, q)\}$,
- $\lambda_{\alpha,\beta} = c_3$ for $(\alpha, \beta) \in \{(b, c), (b, r), (a, c), (a, r), (p, c), (p, r), (q, c), (q, r)\}$,
- $\lambda_{\alpha,\beta} = c_1 c_2$ for $(\alpha, \beta) \in \{(c, d), (c, s), (r, d), (r, s)\}$,
- $\lambda_{\alpha,\beta} = c_1 c_3$ for $(\alpha, \beta) \in \{(b, d), (b, s), (q, d), (q, s)\}$,
- $\lambda_{\alpha,\beta} = c_2 c_3$ for $(\alpha, \beta) \in \{(a, d), (a, s), (p, d), (p, s)\}$.

Here we denoted $c_i = \sigma_i \left(\frac{-1}{\Phi(f_1, f_{-1}) \cdot \Phi(f_2, f_{-2})} \right) = \sigma_i \left(\frac{-1}{4\rho^2} \right)$.

Then in the formulas in Section 6 we can take:

- $m(g) = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}$ is an 8×8 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)} : (dabcspqr)$,
- $m(h_1) = \begin{pmatrix} 0 & X_1^{-1} \\ \frac{1}{c_2 c_3} \cdot X_1 & 0 \end{pmatrix}$ is an 8×8 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)} : (dabcpsrq)$,
- $m(h_2) = \begin{pmatrix} 0 & X_2^{-1} \\ \frac{1}{c_1 c_3} \cdot X_2 & 0 \end{pmatrix}$ is an 8×8 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)} : (dabcqrsp)$,
- $m(h_3) = \begin{pmatrix} 0 & X_3^{-1} \\ \frac{1}{c_1 c_2} \cdot X_3 & 0 \end{pmatrix}$ is an 8×8 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)} : (dabcqrps)$,
- $m(g^k \cdot h_1^{a_1} h_2^{a_2} h_3^{a_3}) = m(g)^k \cdot g^k (m(h_1)^{a_1} \cdot m(h_2)^{a_2} \cdot m(h_3)^{a_3})$, where $0 \leq a_i \leq 1$, $k \geq 0$.

Here we denoted $G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 c_3 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_2 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ 0 & 0 & c_1 c_3 & 0 \\ 0 & 0 & 0 & c_1 \end{pmatrix}$

and $X_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & c_1 c_2 \end{pmatrix}$.

Note that $m(h_i) \cdot m(h_j) = m(h_j) \cdot m(h_i)$, $m(h_i)^2 = \frac{c_i}{c_1 c_2 c_3}$, $m(g)^3 = 1$ and $m(gh_i g^{-1}) = m(g) \cdot g(m(h_i)) \cdot m(g)^{-1}$.

This implies that the class of D in $H^2(S, F^*)$ is represented by the 2-cocycle $\lambda : S \times S \rightarrow F^*$ such that $\lambda(h_1^{a_1} h_2^{a_2} h_3^{a_3}, h_1^{b_1} h_2^{b_2} h_3^{b_3}) = (c_2 c_3)^{x_1} \cdot (c_1 c_3)^{x_2} \cdot (c_1 c_2)^{x_3}$ and $\lambda(g^k h, g^l h') = g^{k+l}(\lambda(g^{-l} h g^l, h'))$, where $0 \leq a_i \leq 1$, $0 \leq b_i \leq 1$, $x_i = 1$ if $a_i = b_i = 1$ and 0 otherwise, and h, h' are elements of the subgroup $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \subset S$ generated by h_1, h_2, h_3 .

Since $c_i c_j = \left(\frac{1}{4 \cdot \sigma_i(\rho) \sigma_j(\rho)} \right)^2$ is a square in L^* , we conclude that λ is a coboundary. Namely, the required morphism $c : S \rightarrow F^*$ (whose coboundary is λ) can be defined as follows:

$$c(g^k \cdot h_1^{a_1} h_2^{a_2} h_3^{a_3}) = g^k ((\sqrt{c_2 c_3})^{a_1} \cdot (\sqrt{c_1 c_3})^{a_2} \cdot (\sqrt{c_1 c_2})^{a_3}),$$

where $0 \leq a_i \leq 1$, $k \geq 0$. Note that $c(gh_i g^{-1}) = g(c(h_i))$. So, the class of D in $H^2(S, F^*)$ vanishes. Hence $D \cong \mathbb{Q}$.

So, in this example $\text{End}(KS(X))_{\mathbb{Q}} \cong \text{Mat}_{256 \times 256}(\mathbb{Q})$.

(2) Now let us consider the case $m = 6$. The root system is of type D_3 : $R_0 = \{\pm \epsilon_p \pm \epsilon_q \mid p, q = 1, 2, 3\}$ with basis $B_0 = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_2 + \epsilon_3\}$. Hence $B_i =$

$\{\epsilon_1 \otimes_{L, \sigma_i} \Gamma_1 - \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2, \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2 - \epsilon_3 \otimes_{L, \sigma_i} \Gamma_3, \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2 + \epsilon_3 \otimes_{L, \sigma_i} \Gamma_3\}$, $1 \leq i \leq 3$, and the Weyl group is generated by sign inversions in front of two of $\epsilon_1, \epsilon_2, \epsilon_3$ and by all possible permutations of $\epsilon_1, \epsilon_2, \epsilon_3$.

The restriction of the spin representation of $\mathfrak{so}(\phi) \otimes_k F$ in $C^+(V \otimes_k F)$ to $\mathfrak{g} \otimes_k F = \text{Res}_{L/k}(\mathfrak{so}(\Phi)) \otimes_k F$ is isomorphic over F to the sum of the exterior tensor products of semi-spin representations (in all possible combinations) each with multiplicity 2^8 : $C^+(V \otimes_k F) \cong \bigoplus_{\alpha_1, \alpha_2, \alpha_3 \in \{\pm\}} 2^8 \cdot (\rho^{\alpha_1} \boxtimes \rho^{\alpha_2} \boxtimes \rho^{\alpha_3})$. Hence the set Ω of highest weights consists of the elements $\omega_{\alpha_1, \alpha_2, \alpha_3} = \frac{1}{2} \cdot \sum_{i=1}^3 (\epsilon_1 \otimes_{L, \sigma_i} \Gamma_1 + \epsilon_2 \otimes_{L, \sigma_i} \Gamma_2 + \alpha_i \cdot \epsilon_3 \otimes_{L, \sigma_i} \Gamma_3)$ for various $\alpha_i \in \{\pm 1\}$.

Note that $g(\omega_{\alpha_1, \alpha_2, \alpha_3}) = \omega_{\alpha_3, \alpha_1, \alpha_2}$ and $h_i(\omega_{\alpha_1, \alpha_2, \alpha_3}) = \omega_{-\alpha_1, -\alpha_2, -\alpha_3}$. So, $\Omega = \Omega_1 \cup \Omega_2$ has two S -orbits: $\Omega_1 = \{\omega_{+, +, +}, \omega_{-, -, -}\}$ and $\Omega_2 = \{\omega_{+, +, -}, \omega_{+, -, +}, \omega_{-, +, +}, \omega_{-, -, +}, \omega_{-, +, -}, \omega_{+, -, -}\}$.

Hence over $k = \mathbb{Q}$ we have: $C^+(V) \cong U^{\oplus \mu} \oplus V^{\oplus \nu}$ as \mathfrak{g} -modules, where U and V are not isomorphic as representations of $\mathfrak{g} = \text{Res}_{L/k}(\mathfrak{so}(\Phi))$. $U \otimes_k F$ splits into $\frac{2^8}{\mu}$ copies of $\rho^+ \boxtimes \rho^+ \boxtimes \rho^+$ and $\frac{2^8}{\mu}$ copies of $\rho^- \boxtimes \rho^- \boxtimes \rho^-$, while $V \otimes_k F$ splits into $\frac{2^8}{\nu}$ copies of $\rho^{\alpha_1} \boxtimes \rho^{\alpha_2} \boxtimes \rho^{\alpha_3}$ with other α_i 's.

In order to estimate multiplicities μ and ν , let us consider

$$f_{1, \dots, 1} = f_1 \cdot \dots \cdot f_l = q \cdot \prod_{i=1}^l \left(e_i + \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right)$$

(we use notation introduced above), where $q \in F$ is such that $\sigma(q) = \pm q$ for any $\sigma \in S = \text{Gal}(F/k)$. In our case

$$f_{1, \dots, 1} = q \cdot (e_1 + \sqrt{-1} \cdot \sqrt{\rho} \cdot e_6) \cdot (e_2 + \sqrt{-1} \cdot \sqrt{\rho} \cdot e_5) \cdot (e_3 + \sqrt{-1} \cdot e_4).$$

Hence the stabilizer of (the line in $C(V \otimes_L F)$ generated by) $f_{1, \dots, 1}$ consists of the elements g^k . Since the stabilizer of $\omega_{+, +, +} \in \Omega$ as a subgroup of S is generated by elements $g, h_1 h_2, h_1 h_3, h_2 h_3$, we conclude that $n_{+, +, +} = 4$. Since the stabilizer of $\omega_{+, +, -} \in \Omega$ has 4 elements: Id and $h_1 h_2, h_1 h_3, h_2 h_3$, we conclude that $n_{+, +, -} = 4$ as well. The same computation as in the case $m = 5$ above shows that $\frac{2^8}{\mu} = \frac{2^8}{\nu} = 1$, i.e. $\mu = \nu = 256$, and the division algebras $D_1 = \text{End}_{\mathfrak{g}}(U)$ and $D_2 = \text{End}_{\mathfrak{g}}(V)$ are fields, i.e. coincide with their centers.

According to Section 6.2, the center C_1 of D_1 is the subfield of F fixed by the stabilizer of $\omega_{+, +, +} \in \Omega$, i.e. $D_1 = C_1 \cong k(\sqrt{-1})$. Similarly, the center C_2 of D_2 is the subfield of F fixed by the stabilizer of $\omega_{+, +, -} \in \Omega$, i.e. $D_2 = C_2 \cong k(\sqrt{-1}, \rho)$.

So, in this example $\text{End}(KS(X))_{\mathbb{Q}} \cong \text{Mat}_{256 \times 256}(\mathbb{Q}(\sqrt{-1})) \times \text{Mat}_{256 \times 256}(\mathbb{Q}(\sqrt{-1}, \rho))$.

(3) Let us modify the first example above. Consider the same number ρ and the same totally real cubic field $E = L = k(\rho)$, but a different quadratic form

$$\Phi = -(a + \rho) \cdot X_1^2 - (a + \rho) \cdot X_2^2 - X_3^2 - X_4^2 - X_5^2,$$

where a is a fixed rational number between 0 and $-\rho$: $0 < a < -\rho$. As above, these quadratic form and totally real field correspond to a $K3$ surface X ([9]). Assume that $1 + 3a - a^3 > 0$ is not a square of a rational number.

Let $F = k \left(\sqrt{a+\rho}, \sqrt{a+\frac{1}{1-\rho}}, \sqrt{a+1-\frac{1}{\rho}}, \sqrt{-1} \right)$ be our choice of a splitting field. Note that $L \subset F$ and $\sqrt{a+\rho} \cdot \sqrt{a+\frac{1}{1-\rho}} \cdot \sqrt{a+1-\frac{1}{\rho}} = \sqrt{-1-3a+a^3}$. Then

$$S = \text{Gal}(F/k) \cong \mathbb{Z}/2\mathbb{Z} \oplus G,$$

where G is the group isomorphic to the Galois group of the splitting field from the first example above, i.e. G is a noncommutative group extension of $\mathbb{Z}/3\mathbb{Z} \cong \text{Gal}(L/k)$ by $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$. Let g be a generator of $\mathbb{Z}/3\mathbb{Z}$ such that $g(\sqrt{a+\rho}) = \sqrt{a+\frac{1}{1-\rho}}$, $g\left(\sqrt{a+\frac{1}{1-\rho}}\right) = \sqrt{a+1-\frac{1}{\rho}}$, $g\left(\sqrt{a+1-\frac{1}{\rho}}\right) = \sqrt{a+\rho}$, $g(\sqrt{-1}) = \sqrt{-1}$. Let h_1, h_2, h_3 be the generators of $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$ and h_0 be the generator of the first factor $\mathbb{Z}/2\mathbb{Z}$ in S above such that each h_i , $0 \leq i \leq 3$ multiplies by -1 the i -th square root among $\sqrt{-1}, \sqrt{a+\rho}, \sqrt{a+\frac{1}{1-\rho}}, \sqrt{a+1-\frac{1}{\rho}}$ and does not change the others. We also assume that $h_i|_L = Id$, $0 \leq i \leq 3$.

There are 3 field embeddings $L \hookrightarrow F$: $\sigma_1 = Id$, $\sigma_2 = g|_L$ and $\sigma_3 = g^2|_L$. Then $\sqrt{\sigma_1(d_1)} = \sqrt{\sigma_1(d_2)} = \sqrt{-1} \cdot \sqrt{a+\rho}$, $\sqrt{\sigma_2(d_1)} = \sqrt{\sigma_2(d_2)} = \sqrt{-1} \cdot \sqrt{a+\frac{1}{1-\rho}}$, $\sqrt{\sigma_3(d_1)} = \sqrt{\sigma_3(d_2)} = \sqrt{-1} \cdot \sqrt{a+1-\frac{1}{\rho}}$, $\sqrt{\sigma_1(d_3)} = \sqrt{\sigma_2(d_3)} = \sqrt{\sigma_3(d_3)} = \sqrt{-1}$, $\sqrt{-\sigma_i(d_4)} = \sqrt{-\sigma_i(d_5)} = 1$ for any $i = 1, 2, 3$. Hence $\otimes_{L,\sigma_1} \Gamma_1 = \otimes_{L,\sigma_1} \Gamma_2 = \sqrt{-1} \cdot \sqrt{a+\rho}$, $\otimes_{L,\sigma_2} \Gamma_1 = \otimes_{L,\sigma_2} \Gamma_2 = \sqrt{-1} \cdot \sqrt{a+\frac{1}{1-\rho}}$, $\otimes_{L,\sigma_3} \Gamma_1 = \otimes_{L,\sigma_3} \Gamma_2 = \sqrt{-1} \cdot \sqrt{a+1-\frac{1}{\rho}}$.

As in the first example above, the root system is of type B_2 : $R_0 = \{\pm\epsilon_p, \pm\epsilon_p \pm \epsilon_q \mid p, q = 1, 2\}$ with basis $B_0 = \{\epsilon_1 - \epsilon_2, \epsilon_2\}$. Hence $B_i = \{\epsilon_1 \otimes_{L,\sigma_i} \Gamma_1 - \epsilon_2 \otimes_{L,\sigma_i} \Gamma_2, \epsilon_2 \otimes_{L,\sigma_i} \Gamma_2\}$, $1 \leq i \leq 3$. The restriction of the spin representation of $\mathfrak{so}(\phi) \otimes_k F$ in $C^+(V \otimes_k F)$ to $\mathfrak{g} \otimes_k F = \text{Res}_{L/k}(\mathfrak{so}(\Phi)) \otimes_k F$ is isomorphic over F to 2^8 copies of the exterior tensor product $\rho^0 \boxtimes \rho^0 \boxtimes \rho^0$ of the irreducible spin representation of $\mathfrak{so}(\Phi) \otimes_L F$. Hence over $k = \mathbb{Q}$ the restriction of the spin representation of $\mathfrak{so}(\phi)$ in $C^+(V)$ to $\mathfrak{g} = \text{Res}_{L/k}(\mathfrak{so}(\Phi)) \subset \mathfrak{so}(\phi)$ is one single irreducible representation with multiplicity μ which splits over F into $\frac{2^8}{\mu}$ copies of $\rho^0 \boxtimes \rho^0 \boxtimes \rho^0$: $C^+(V) \cong U^{\oplus \mu}$.

In order to estimate $\frac{2^8}{\mu}$ (which divides n_0), let us consider

$$f_{1,\dots,1,1} = f_1 \cdot \dots \cdot f_l \cdot (1 + f_0) = q \cdot \prod_{i=1}^l \left(e_i + \frac{\sqrt{d_i}}{\sqrt{-d_{m-i+1}}} \cdot e_{m-i+1} \right) \cdot \left(1 + \frac{1}{\sqrt{d_{l+1}}} \cdot e_{l+1} \right)$$

(we use notation as above), where $q \in F$ is such that $\sigma(q) = \pm q$ for any $\sigma \in S = \text{Gal}(F/k)$. In our case

$$f_{1,\dots,1,1} = q \cdot (e_1 + \sqrt{-1} \cdot \sqrt{a+\rho} \cdot e_5) \cdot (e_2 + \sqrt{-1} \cdot \sqrt{a+\rho} \cdot e_4) \cdot (1 - \sqrt{-1} \cdot e_3).$$

Hence the stabilizer of (the line in $C(V \otimes_L F)$ generated by) $f_{1,\dots,1,1}$ consists of the elements g^k , i.e. has order 3. Since $\text{Gal}(F/k)$ has 48 elements total, we find that $n_0 = 16$. Hence either $\frac{2^8}{\mu} = 1$ or $\frac{2^8}{\mu} = 2$ or $\frac{2^8}{\mu} = 4$ or $\frac{2^8}{\mu} = 8$ or $\frac{2^8}{\mu} = 16$. In the first case, $\rho^0 \boxtimes \rho^0 \boxtimes \rho^0$ is already defined over \mathbb{Q} and $\mu = 2^8$, while in the other cases $\mu = 2^7, \mu = 2^6, \mu = 2^5$ and $\mu = 2^4$ respectively.

Hence in this case $\text{End}(KS(X))_{\mathbb{Q}} \cong \text{Mat}_{\mu \times \mu}(D)$, where $D = \text{End}_{\mathfrak{g}}(U)$ is a division algebra. Let us compute the cohomological invariant of D . In our case

$$\begin{aligned} W \otimes_k F = & V_{(1,1,1)} \oplus V_{(1',1,1)} \oplus V_{(1,1',1)} \oplus V_{(1,1,1')} \oplus V_{(1',1',1')} \oplus V_{(1,1',1')} \oplus V_{(1',1,1')} \oplus V_{(1',1',1)} \oplus \\ & \oplus V_{(2,2,2)} \oplus V_{(2',2,2)} \oplus V_{(2,2',2)} \oplus V_{(2,2,2')} \oplus V_{(2',2',2')} \oplus V_{(2,2',2')} \oplus V_{(2',2,2')} \oplus V_{(2',2',2)}, \end{aligned}$$

where $V_{(p_1,p_2,p_3)} = S_{p_1}^1 \otimes_F S_{p_2}^2 \otimes_F S_{p_3}^3$ in the notation of Section 5.2 and the values $1, 1', 2, 2'$ of p_i correspond to the indices $(\alpha_1, \alpha_2, \gamma)$ of ideals $I_{\alpha_1, \alpha_2, \gamma}$ as follows: $1 = (+++)$, $1' = (- - +)$, $2 = (+ + -)$, $2' = (- - -)$.

Let us denote $\bar{1} = 1', \bar{1}' = 1, \bar{2} = 2', \bar{2}' = 2$ and $\tilde{1} = 2', \tilde{1}' = 2, \tilde{2} = 1', \tilde{2}' = 1$. Then $g(V_{(p_1,p_2,p_3)}) = V_{(p_3,p_1,p_2)}$, $h_0(V_{(p_1,p_2,p_3)}) = V_{(\tilde{p}_1,\tilde{p}_2,\tilde{p}_3)}$ and $h_i(V_{(p_1,p_2,p_3)}) = V_{(q_1,q_2,q_3)}$, $1 \leq i \leq 3$, where $q_i = \bar{p}_i$ and $q_j = p_j$ for $j \neq i$.

Let us denote $a = (1', 1, 1)$, $b = (1, 1', 1)$, $c = (1, 1, 1')$, $d = (1, 1, 1)$, $a' = (1, 1', 1')$, $b' = (1', 1, 1')$, $c' = (1', 1', 1)$, $d' = (1', 1', 1')$, $p = (2', 2, 2)$, $q = (2, 2', 2)$, $r = (2, 2, 2')$, $s = (2, 2, 2)$, $p' = (2, 2', 2')$, $q' = (2', 2, 2')$, $r' = (2', 2', 2)$, $s' = (2', 2', 2')$. Consider the set of indices $T = \{d, a, b, c, d', a', b', c', s, p, q, r, s', p', q', r'\}$ and the morphism $t: T \rightarrow T, s \mapsto d, p \mapsto a, q \mapsto b, r \mapsto c, s' \mapsto d', p' \mapsto a', q' \mapsto b', r' \mapsto c'$ and $x \mapsto x$ for all other $x \in T$.

Then using formulas from Section 6 we can choose coefficients $\lambda_{\alpha,\beta} = \prod_{i=1}^r \lambda_{\alpha^i, \beta^i} \in F^*$ as follows:

- $\lambda_{d,x} = \lambda_{s,x} = \lambda_{x,x} = 1$, $\lambda_{d',d} = c_1 c_2 c_3$ and $\lambda_{x,y} = \lambda_{t(x),t(y)}$ for any $x, y \in T$,
- $\lambda_{\alpha,\beta} = 1$ for $(\alpha, \beta) \in \{(a, d'), (a, b'), (a, c'), (b, d'), (b, a'), (b, c')\}$,
- $\lambda_{\alpha,\beta} = 1$ for $(\alpha, \beta) \in \{(c, d'), (c, b'), (c, a'), (a', d'), (b', d'), (c', d')\}$,
- $\lambda_{\alpha,\beta} = c_1$ for $(\alpha, \beta) \in \{(a, a'), (a, d), (a, b), (a, c), (d', a'), (b', c), (b', a'), (c', b), (c', a')\}$,
- $\lambda_{\alpha,\beta} = c_2$ for $(\alpha, \beta) \in \{(b, b'), (b, d), (b, a), (b, c), (d', b'), (a', c), (a', b'), (c', a), (c', b')\}$,
- $\lambda_{\alpha,\beta} = c_3$ for $(\alpha, \beta) \in \{(c, c'), (c, d), (c, b), (c, a), (d', c'), (a', b), (a', c'), (b', a), (b', c')\}$,
- $\lambda_{\alpha,\beta} = c_1 c_2$ for $(\alpha, \beta) \in \{(d', c), (c', c), (c', d)\}$,
- $\lambda_{\alpha,\beta} = c_1 c_3$ for $(\alpha, \beta) \in \{(d', b), (b', b), (b', d)\}$,
- $\lambda_{\alpha,\beta} = c_2 c_3$ for $(\alpha, \beta) \in \{(d', a), (a', a), (a', d)\}$.

Here we denoted $c_i = \sigma_i \left(\frac{-1}{\Phi(f_1, f_{-1}) \cdot \Phi(f_2, f_{-2})} \right) = \sigma_i \left(\frac{-1}{4(a+\rho)^2} \right)$.

Then in the formulas in Section 6 we can take:

- $m(g) = \begin{pmatrix} G & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G \end{pmatrix}$ is a 16×16 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)}$: $(abcd' a' b' c' spqr s' p' q' r')$,
- $m(h_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{c_1} \cdot 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{c_1} \cdot 1 & 0 \end{pmatrix}$ is a 16×16 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)}$: $(da'bcad' c' b' sp'qrps' r' q')$,
- $m(h_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{c_2} \cdot 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{c_2} \cdot 1 & 0 \end{pmatrix}$ is a 16×16 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)}$: $(dab'cbc' d' a' spq'rqr' s' p')$,
- $m(h_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{c_3} \cdot 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{c_3} \cdot 1 & 0 \end{pmatrix}$ is a 16×16 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)}$: $(dabc'cb' a' d' spqr' r' q' p' s')$,
- $m(h_0) = \begin{pmatrix} 0 & X_0^{-1} & 0 & 0 \\ \frac{1}{c_1 c_2 c_3} \cdot X_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c_1 c_2 c_3} \cdot X_0 \\ 0 & 0 & X_0^{-1} & 0 \end{pmatrix}$ is a 16×16 matrix whose rows and columns are numbered according to the following sequence of indices of $V_{(p_1, p_2, p_3)}$: $(dabcs' p' q' r' d' a' b' c' spqr)$,
- $m(g^k \cdot h_0^{a_0} h_1^{a_1} h_2^{a_2} h_3^{a_3}) = m(g)^k \cdot g^k (m(h_0)^{a_0} \cdot m(h_1)^{a_1} \cdot m(h_2)^{a_2} \cdot m(h_3)^{a_3})$, where $0 \leq a_i \leq 1, k \geq 0$.

Here we denoted $G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (in the definitions of $m(h_i)$)

and $X_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_3 \end{pmatrix}$.

Note that $m(h_i) \cdot m(h_j) = m(h_j) \cdot m(h_i)$, $m(h_i)^2 = \frac{1}{c_i}$, $1 \leq i \leq 3$, $m(h_0)^2 = \frac{1}{c_1 c_2 c_3}$, $m(g)^3 = 1$ and $m(gh_i g^{-1}) = m(g) \cdot g(m(h_i)) \cdot m(g)^{-1}$.

This implies that the class of D in $H^2(S, F^*)$ is represented by the 2-cocycle $\lambda: S \times S \rightarrow F^*$ such that

$$\lambda(h_0^{a_0} h_1^{a_1} h_2^{a_2} h_3^{a_3}, h_0^{b_0} h_1^{b_1} h_2^{b_2} h_3^{b_3}) = (c_1 c_2 c_3)^{x_0} \cdot (c_2 c_3)^{x_1} \cdot (c_1 c_3)^{x_2} \cdot (c_1 c_2)^{x_3}$$

and $\lambda(g^k h, g^l h') = g^{k+l}(\lambda(g^{-l} h g^l, h'))$, where $0 \leq a_i \leq 1$, $0 \leq b_i \leq 1$, $x_i = 1$ if $a_i = b_i = 1$ and 0 otherwise, and h, h' are elements of the subgroup $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \subset S$ generated by h_0, h_1, h_2, h_3 .

Let us multiply λ by the inverse of the coboundary of the 1-cochain given by the morphism $c: S \rightarrow F^*$ such that

$$c(g^k \cdot h_0^{a_0} h_1^{a_1} h_2^{a_2} h_3^{a_3}) = g^k ((\sqrt{c_1 c_2 c_3})^{a_0} \cdot (\sqrt{c_1})^{a_1} \cdot (\sqrt{c_2})^{a_2} \cdot (\sqrt{c_3})^{a_3}),$$

where $0 \leq a_i \leq 1$, $k \geq 0$. Note that $c(gh_i g^{-1}) = g(c(h_i))$.

This changes λ to a 2-cocycle $\lambda': S \times S \rightarrow F^*$ such that

$$\lambda'(g^k \cdot h_0^{a_0} h_1^{a_1} h_2^{a_2} h_3^{a_3}, g^l \cdot h_0^{b_0} h_1^{b_1} h_2^{b_2} h_3^{b_3}) = (-1)^{a_0 \cdot (b_0 + b_1 + b_2 + b_3)},$$

where $0 \leq a_i \leq 1$, $0 \leq b_i \leq 1$.

Let $H \subset S$ be the subgroup generated by $g, h_1 h_2, h_1 h_3, h_2 h_3$ and

$$F^H = k \left(\sqrt{-1}, \sqrt{(a + \rho)(a + \frac{1}{1 - \rho})(a + 1 - \frac{1}{\rho})} \right) = k(\sqrt{-1}, \sqrt{-1 - 3a + a^3})$$

be the corresponding fixed subfield of F . Denote the generators of $\text{Gal}(F^H/k) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ by h_0 and $h = h_1 h_2 h_3$.

We see that the class of D in $H^2(S, F^*)$ is the image under the inflation homomorphism $H^2(\text{Gal}(F^H/k), (F^H)^*) \rightarrow H^2(S, F^*)$ of a class represented by the 2-cocycle $\lambda'': \text{Gal}(F^H/k) \times \text{Gal}(F^H/k) \rightarrow k(\sqrt{-1}, \sqrt{-1 - 3a + a^3})^*$ such that $\lambda''(h_0, h_0) = \lambda''(h_0, h) = -\lambda''(h, h_0) = -\lambda''(h, h) = -1$. Multiplying it by the coboundary of the 1-cochain given by the morphism $c: \text{Gal}(F^H/k) \rightarrow (F^H)^*$ such that $c(h) = c(h_0) = \sqrt{-1}$, $c(hh_0) = 1$, we obtain a 2-cocycle (also denoted by λ'') with the property $\lambda''(h_0 h, -) = \lambda''(-, h_0 h) = 1$ and $\lambda''(h_0, h_0) = 1$. Note that $(F^H)^{<h_0 h>} = k(\sqrt{1 + 3a - a^3})$ is a totally real quadratic field with Galois group $\mathbb{Z}/2\mathbb{Z}$ with generator 1.

This means that the cohomological class of D can be obtained via the inflation homomorphism from the class in $H^2(\text{Gal}(k(\sqrt{1 + 3a - a^3})/k), k(\sqrt{1 + 3a - a^3})^*)$ of the 2-cocycle $\lambda_0: \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow k(\sqrt{1 + 3a - a^3})^*$ such that $\lambda_0(1, 1) = -1$.

Hence D is a quaternion algebra over $\mathbb{Q} = k$ of degree $\deg(D) = 2$ split over $\mathbb{Q}(\sqrt{1 + 3a - a^3})$ with 4 generators over \mathbb{Q} : $1, i, j, k$ such that $i^2 = j^2 = 1 + 3a - a^3$, $k = ij = -ji$. In other words, $D = (1 + 3a - a^3, 1 + 3a - a^3)_{\mathbb{Q}}$.

So, in this example $\text{End}(KS(X))_{\mathbb{Q}} \cong \text{Mat}_{128 \times 128}((1 + 3a - a^3, 1 + 3a - a^3)_{\mathbb{Q}})$.

(4) If in the previous example we take

$$\Phi = -(b \cdot \rho) \cdot X_1^2 - (b \cdot \rho) \cdot X_2^2 - X_3^2 - X_4^2 - X_5^2,$$

where $b > 0$ is a rational number which is not a square of another rational number, then the same computation as above gives:

$$\text{End}(KS(X))_{\mathbb{Q}} \cong \text{Mat}_{128 \times 128}((b, b)_{\mathbb{Q}})$$

for the corresponding $K3$ surface X .

8 Acknowledgement.

We thank Yuri Zarhin for suggesting this problem and for pointing out an error in the Example section of the previous version. Many of our constructions were influenced by papers [12], [13] and [14], where Bert van Geemen studies endomorphism algebras of Kuga-Satake varieties in more special cases.

References

- [1] N. Bourbaki, *Elements de mathematique. Groupes et algebres de Lie. Chapitre VIII: Algebres de Lie semi-simples deployees*, Hermann, Paris, 1975.
- [2] C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, 1954.
- [3] B. Conrad, *Reductive group schemes*, SGA3 summer school, <http://math.stanford.edu/~conrad/papers/luminysga3.pdf> (2011), 1–271.
- [4] W. Fulton and J. Harris, *Representation theory. A first course*, Graduate Texts in Mathematics, vol. 129, Springer, 1991.
- [5] N. Jacobson, *Basic algebra II, Second edition*, W.H. Freeman and company, 1985.
- [6] H. Kim, *Recent results on arithmetic of the simplest cubic fields*, Trends in Mathematics **1** (1998), 46–51.
- [7] M. Kuga and I. Satake, *Abelian varieties attached to polarized K3-surfaces*, Mathematische Annalen **169** (1967), 239–242.

- [8] A. Kuznetsov, *Course of algebra*, IUM 2002-2003 course (2003).
- [9] E. Mayanskiy, *Hermitian forms of K3 type*, ArXiv preprint (2012), 1–9.
- [10] D. Morrison, *The Kuga-Satake Variety of an Abelian Surface*, Journal of Algebra **92** (1985), 454–476.
- [11] U. Schlickewei, *The Hodge conjecture for self-products of certain K3 surfaces*, Journal of Algebra **324** (2010), 507–529.
- [12] B. van Geemen, *Kuga-Satake varieties and the Hodge conjecture*, In: *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, NATO Science Series C: Mathematical and Physical Sciences, vol. 548, Kluwer Acad. Publ, 2000.
- [13] ———, *Half twists of Hodge structures of CM-type*, Journal of the Mathematical Society of Japan **53** (2001), 813–833.
- [14] ———, *Real multiplication on K3 surfaces and Kuga-Satake varieties*, Michigan Mathematical Journal **56** (2008), 375–399.
- [15] Y. Zarhin, *Hodge groups of K3 surfaces*, Journal für die Reine und Angewandte Mathematik **341** (1983), 193–220.